

Math 112 Problem Set 2

Due Feb. 7, 2018

February 5, 2018

Core problems:

1. Let $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ be bounded below. Define

- $A + B = \{x + y \mid x \in A, y \in B\}$
- $A \cdot B = \{xy \mid x \in A, y \in B\}$

- (a) Is it true that $\inf(A + B) = \inf A + \inf B$?
- (b) Is it true that $\inf(A \cdot B) = (\inf A)(\inf B)$?

For each, either prove the equality or find a counter-example.

2. Let $\{x_n\}_n$ be a Cauchy sequence. Suppose that, for every $\epsilon > 0$, there is some $n > \frac{1}{\epsilon}$ such that $|x_n| < \epsilon$. Prove that $x_n \rightarrow 0$. (1.4, #4)

3. For nonempty sets $A, B \subset \mathbb{R}$, determine which of the following are true. Prove the true statements and give a counterexample for those that are false:

- (a) $\sup(A \cap B) \leq \inf\{\sup(A), \sup(B)\}$
- (b) $\sup(A \cap B) = \inf\{\sup(A), \sup(B)\}$
- (c) $\sup(A \cup B) \geq \sup\{\sup(A), \sup(B)\}$
- (d) $\sup(A \cup B) = \sup\{\sup(A), \sup(B)\}$

(1, #8)

4. 1.3, #1

5. 1, #4

6. 1, #5

7. Let A be a set bounded ~~above~~ below and set $-A = \{-x \mid x \in A\}$. Show that

$$\inf A = -\sup(-A)$$

8. Complete Proposition 1.2.20 by proving the following by induction: if $x_n = \sum_{k=1}^n \frac{1}{k}$, then

$$x_{2^n} \geq 1 + \frac{n}{2}$$

9. Let $\{x_n\}_n$ be a bounded sequence of real numbers. Show that $\{\frac{x_n}{n}\}_n$ converges. Find a (unbounded) sequence $\{y_n\}_n$ so that $\{\frac{y_n}{n}\}_n$ does not converge and prove that.

Niche problems:

1. Suppose that $\{a_n\}_n$ and $\{b_n\}_n$ are sequences such that $\lim_{n \rightarrow \infty} b_n - a_n = 0$ and $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$. Show that $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$ and that this limit is the only $x \in \mathbb{R}$ such that $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$.

This is actually another formulation of the completeness of the real numbers! (This means that you'll have to use completeness to show it.)

2. Say an ordered field \mathbb{F} has the least upper bound property iff any $S \subset \mathbb{F}$ bounded above has a least upper bound (as in Theorem 1.3.4). Show that \mathbb{F} must satisfy the Monotone Sequence Property (1.2.8)
3. This is the first in a sequence of Niche problems that will explore the formal construction of the real numbers. As it progresses, feel free to use previous parts of it if you want to jump in at a later stage.

Given $A \subset \mathbb{Q}$, call A a *Dedekind cut* iff

- if $x \in A$ and $y < x$, then $y \in A$ (downward closed)
- if $x \in A$, then there is $y \in A$ such that $y > x$ (no greatest element)
- A is not empty and is not \mathbb{Q}

Set $\mathcal{D} = \{A \subset \mathbb{Q} \mid A \text{ is a Dedekind cut}\}$. Define $f : \mathcal{D} \rightarrow \mathbb{R}$ by

$$f(A) = \sup A$$

Show that f is a bijection.

Doc Brown problems:

1. 1.5, #1
2. 1.6, #5
3. 2.1, #1

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.