Math 112 Problem Set 11

Due Wednesday, April 25, 2018

April 24, 2018

Core problems:

1. Prove that a sequence $f_k : A \to \mathbb{R}^m$ converges pointwise iff, for each $x \in A$, $\{f_k(x)\}_k$ is a Cauchy sequence. (5, #6)

2. Does pointwise convergence of continuous functions to a continuous limit imply uniform convergence on that set? (5, #8)

3. From the sequences in (7) and (8) last week, choose 5 and determine whether they can be integrated and/or differentiated. (5, #10)

4. Let $f : [0,1] \to \mathbb{R}$ be continuous and one-to-one. Show that f is either increasing or decreasing. (5, #25)

5. Find functions $f, g : [0,1] \to \mathbb{R}$ such that

$$2\|f\|^2 + 2\|g\|^2 \neq \|f+g\|^2 + \|f-g\|^2$$

where $\|h\| = \sup\{|h(x)| \mid x \in [0,1]\}$ is the sup norm. This means that the parallelogram law fails and that the sup norm does not come from an inner product (5, #48, also see Chapter 1.7 and 1, #12)

6. Let $f_n : [0,1] \to \mathbb{R}$ be monotone increasing and continuous such that, for every $x \in [0,1]$, $F(x) = \sum_{n=0}^{\infty} f_n(x)$ converges (so the sequence of functions converges pointwise). Show that F is continuous. (5, #69)

7. We know that continuous functions are integrable, but we prove something more: Let $f : [a,b] \to \mathbb{R}$ be monotone (either increasing or decreasing). Show that it is integrable.

Niche problems:

1. We've shown that exp_n converges absolutely and uniformly to exp on compact sets. Show that

   - exp_n converges pointwise to exp on $\mathbb{R}$.
   - exp_n *does not* converge uniformly to exp on $\mathbb{R}$.

   Why is this last part not a problem in developing the properties of exp?

2. Define $g : \mathbb{R} \to \mathbb{R}$ by setting $g(x) = |x|$ for $x \in [-\frac{1}{2}, \frac{1}{2}]$ and saying that g has period 1. Define

$$f(x) = \sum_{n=1}^{\infty} \frac{g(4^{n-1}x)}{4^{n-1}}$$

   (a) Sketch g and the first few terms of the sum.

   (b) Use the Weierstrass M test to show that f is continuous.

   (c) Prove that f is differentiable at no point.

3. There are lots of neat results that I don't think we'll have time for. One of them is Taylor's theorem, so it appears here as a Niche problem. If I do get to it, lucky for you!

   Let $f : \mathbb{R} \to \mathbb{R}$ have a $(k+1)$th derivative on all of $\mathbb{R}$. For each $x \in R$, there is a number α between a and x such that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a^k) + \frac{f^{(k+1)}(\alpha)}{(k+1)!}(x-a)^{n+1}$$

4. Measure theory is a continuation of what we've learned that makes more functions integrable by extending the notion of volume to more sets in $\mathbb{R}^n$ (see Math 114). For instance, while we know the function

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

   is not integrable via the (Riemann) integral we have defined, it is integrable using the Lebesgue integral.

   Developing this notion of measure or volume is hard, but the definition of having volume 0 is easier.

   Definition. *Let $A \subset \mathbb{R}$. We say that A* has measure zero *iff, for every $\epsilon > 0$, there are open intervals $(a_n, b_n) \subset R$ for $n \in \mathbb{N}$ such that*

   (a) $A \subset \cup_{n=0}^{\infty}(a_n, b_n)$ and

   (b) $\sum_{n=0}^{\infty}(b_n - a_n) < \epsilon$.

   Show that $\mathbb{Q}$ (the rationals) and C (the Cantor set) both have measure 0.

5. The following exercise is also useful for measure theory: A function $g : [0,1] \to \mathbb{R}$ is called *simple* if we can divide $[0,1]$ up into subintervals on which g is constant, except perhaps at the endpoints. Let $f : [0,1] \to \mathbb{R}$ be continuous and $\epsilon > 0$. Show that there is a simple function g such that

$$\|f - g\| = \sup\{|f(x) - g(x)| \mid x \in [0,1]\} < \epsilon$$

   (5, #39)

Doc Brown problems:

1. Study for your midterms.
2. Live a happy and fulfilling life post-Math 112.

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.