Math 112 Problem Set 10

Due Wednesday, April 18, 2018

April 10, 2018

Core problems: The first couple exercises delve more into the properties of upper and lower sums.

1. Let $P_0 = \{x_0, \ldots, x_n\}$ and $P_1 = \{y_0, \ldots, y_m\}$ both be partitions of $[a, b]$. We say that P_1 *refines* P_0 iff $P_0 \subset P_1$; that is, for each i, there is a j such that $x_i = y_j$. Explain what this means in words and show that, if $f : [a, b] \to \mathbb{R}$ and P_0, P_1 are partitions of $[a, b]$ such that P_1 refines P_0, then

$$L(f, P_0) \leq L(f, P_1) \leq U(f, P_1) \leq L(f, P_0)$$

2. If P_0 and P_1 are partitions of $[a, b]$, show that they have a common refinement P_2. Use this to show that, for any P_0, P_1 that partition $[a, b]$,

$$L(f, P_0) \leq U(f, P_1)$$

3. Validate my shoddy presentation of $\int_0^1 x = \frac{1}{2}$ by showing that, if P_n, Q_n are partitions of $[a, b]$ for $n \in \mathbb{N}$ such that

$$\sup\{L(f, P_n) \mid n \in \mathbb{N}\} = \inf\{U(f, Q_n) \mid n \in \mathbb{N}\}$$

   then f is integrable and find its integral.

4. If $f : [a, b] \to \mathbb{R}$ is continuous such that

   - $f(x) \geq 0$ for all $x \in [a, b]$; and
   - there is $y \in [a, b]$ such that $f(y) > 0$,

   then $\int_a^b f > 0$.

5. Prove Cauchy's Mean Value Theorem: Let f, g be functions that are continuous on $[a, b]$ and differentiable on (a, b) such that $g'(x) \neq 0$ for all $x \in (a, b)$. Show there is a $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

6. Suppose that $f : [a, b] \to \mathbb{R}$ is continuous such that $f(a) = f(b) = 0$ and

$$x^2 f''(x) + 4xf'(x) + 2f(x) \geq 0$$

for all $x \in (a, b)$. Show that $f(x) \leq 0$ for all $x \in [a, b]$. (4, #39)

In the following problems, you can use what you know from calculus for the appropriate problems. Also, you can be briefer with your justifications, but should make it clear why your answer is true in each case.

7. For each of the following sequences $\{f_k\}$, determine wether it converges either pointwise or uniformly on the given domain. If it does, determine the limit.

   (a) $f_k(x) = \frac{\sin x}{k}$ on $\mathbb{R}$
   (b) $f_k(x) = \frac{1}{kx+1}$ on $(0, 1)$
   (c) $f_k(x) = \frac{x}{kx+1}$ on $(0, 1)$
   (d) $f_k(x) = \frac{x}{1+kx^2}$ on $\mathbb{R}$
   (e) $f_k(x) = \left(1, \frac{\cos x}{k^2}\right)$ on $\mathbb{R}$, where $f_k : \mathbb{R} \to \mathbb{R}^2$

   (5, #2)

8. For each of the following sequences $\{g_k\}$, determine wether the series $\sum_{k=1}^{\infty} g_k$ converges either pointwise or uniformly on the given domain. If it does, determine the limit.

   (a) $g_k(x) = \begin{cases} 0 & x \leq k \\ (-1)^k & x > k \end{cases}$ on $\mathbb{R}$

   (b) $g_k(x) = \begin{cases} \frac{1}{k^2} & |x| \leq k \\ \frac{1}{x^2} & |x| > k \end{cases}$ on $\mathbb{R}$

   (c) $g_k(x) = \frac{(-1)^k}{\sqrt{k}} \cos kx$ on $\mathbb{R}$
   (d) $g_k(x) = x^k$ on $(0, 1)$

   (5, #3)

Niche problems:

1. Suppose that we had a mystical, magical function $f : [-1, 1] \to \mathbb{R}$ with the following properties:

   - for all $x \in [-1, 1]$, $f'(x) = f(x)$; and
   - $f(0) = 1$.

   Show that this magical function has an inverse, call it g, and show that it is differentiable with $g'(x) = \frac{1}{x}$. (The interval $[-1, 1]$ here is not important, but specified for definiteness.)

2. We've talked about limits and differentiability, but haven't done your favorite result here: L'Hôpital's Rule. Prove the following weak case of it here: Suppose that f, g are functions differentiable in an open set containing $a \in \mathbb{R}$ such that

   - $f(a) = g(a) = 0$;
   - $\frac{f'(x)}{g'(x)}$ exists

   Show that
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

3. Or maybe weak cases of L'Hôpital's rule aren't for you? Prove the following stronger case: Suppose f, g are differentiable functions in an open set containing $a \in \mathbb{R}$ such that

   - $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ are $\pm\infty$
   - $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists

   Show that
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

   (If you do this stronger one, it counts as doing Niche 2 as well, for a total of 2 problems and 20 points. But you can't do this one AND Niche 2)

4. All good things must come to an end, and here we have reached the end of our semester long series on constructing the real numbers. But wait! We've constructed 2-3 different complete ordered fields: Dedekind cuts, equivalence classes of Cauchy sequences, and (sort of) decimal expansions. Our final problem is to show that these are all the same.

   Let $(\mathbb{F}_1, 0, 1, +, \times)$ and $(\mathbb{F}_2, \bar{0}, \bar{1}, \oplus, \otimes)$ be complete ordered fields. Show that there is a bijection $f : \mathbb{F}_1 \to \mathbb{F}_2$ such that

   - $f(0) = \bar{0}$
   - $f(1) = \bar{1}$
   - $f(x + y) = f(x) \oplus f(y)$
   - $f(x \times y) = f(x) \otimes f(y)$
   - if $A \subset \mathbb{F}_1$ is bounded above, then $f(\sup A) = \sup f(A)$

   Such a map is called an *isomorphism* (of ordered fields). This tells you that the fields are actually the same, even if they might be described in different ways.

   (This problem seems intimidating, but if you start writing it down, it turns out there's only one way to build f. You know the value 0 and 1 have to take. What else does this pin down? Completeness is essential here.)

Doc Brown problems:

1. (5.2, #3)
2. (5.3, #5)
3. (5.5, #1)

For problems from the book, something like 1, #8 refers to #8 from the exercises at the end of Chapter 1, while something like 1.3, #1 refers to #1 from the exercises at the end of section 1.3.