Math 112 Problem Set 1

Due Jan. 31, 2018

January 24, 2018

Core problems:

1. Show that
   (a) $f : S \to T$ is one-to-one iff there is a function $g : T \to S$ such that $g \circ f = \mathrm{I}_S$;
   (b) $f : S \to T$ is onto iff there is a function $h : T \to S$ such that $f \circ h = \mathrm{I}_T$; and
   (c) $f : S \to T$ is a bijection iff there is a map $g : T \to S$ such that $f \circ g = \mathrm{I}_T$ and $g \circ f = \mathrm{I}_S$. Show also that $g = f^{-1}$ is uniquely determined.

   Recall that I_S is the function with domain and range S such that, for all $x \in S$, $I_S(x) = x$. (Intro, #15)

2. Prove that, in an ordered field, if $\sqrt{2}$ is a positive number whose square is 2, then $\sqrt{2} < \frac{3}{2}$. (1.1, #4)

3. Define $\{x_n\}_n$ inductively by $x_n = 0$ and $x_{n+1} = \sqrt{2 + x_n}$ and set $\lambda = \lim_{n\to\infty} x_n$ (Example 1.2.10 shows that this limit exists).
   (a) Show that λ is a root of $\lambda^2 - \lambda - 2 = 0$.
   (b) Find $\lim_{n\to\infty} x_n$. (1.2, #1)

4. Show that $\frac{3^n}{n!}$ converges to 0. (1.2, #2)

5. (Moved to next week)

6. (Moved to next week)

7. (a) Let $x \geq 0$ be a real number such that, for any $\epsilon > 0$, $x \leq \epsilon$. Show that $x = 0$.
   (b) Let $S = (0, 1) \subset \mathbb{R}$. Show that for every $\epsilon > 0$, there is an $x \in S$ such that $x < \epsilon$.

   (1, #3)

8. (Moved to next week)

9. Show that, in $\mathbb{R}$, $\lim_{n\to\infty} x_n = x$ iff $\lim_{n\to\infty}(-x_n) = -x$. Prove that the completeness axiom is equivalent to the statement that every decreasing sequence $x_1 \geq x_2 \geq \ldots$ bounded below converges. YOU DO NOT NEED TO PROVE IT CONVERGES TO A PARTICULAR VALUE(1, #18)

10. Look at Propositions 1.1.2, 1.1.5, and 1.1.6. Pick two items that aren't proved in the book and weren't proved in class and prove them.

Niche problems:

1. (a) Define $\phi : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $\phi(i, j) = j + \frac{1}{2}k(k+1)$ where $k = i + j$. Show that ϕ is a bijection and that it has something to do with [the picture in the book].

   (b) Show that if $A_1, A_2, \ldots$ are countable sets, then so is $\cup_{n=1}^{\infty} A_n$.

   (Intro, #8)

2. Let $x_n = \sqrt{n^2 + 1} - n$. Compute $\lim_{n \to \infty} x_n$. (1.2, #3)

3. For each $x \in \mathbb{R}$ with $x \geq 0$, show that there is a $y \in \mathbb{R}$ such that $y^2 = x$.

4. Prove that $\mathbb{Z}$ is countable.

Doc Brown problems:

1. Find a sequence $\{x_n\}_n$ such that $\limsup_n x_n = 5$ and $\liminf_n x_n = 3$. (1.5, #2)

2. What is the angle between $(3, 2, 2)$ and $(0, 1, 0)$? (1.6, #2)

3. Put the inner product $\langle f, g \rangle = \int_0^1 f(x)g(x)dx$ on $\mathcal{C}([0, 1])$. Verify the Cauchy-Shwarz inequality for $f(x) = 1$ and $g(x) = x$. (1.7, #3)

Note that I have written out the problems that appear in Marsden and Hoffman because it's still shopping period. You shouldn't rely on this in future problem sets.