

PROBLEMS TO PRACTICE BEFORE THE MIDTERM

1. PROBLEM 1

Let \mathbb{R}^2 be the set of (ordered) pairs (a, b) of real numbers. Define an operation $\circ: (\mathbb{R}^2 \setminus \{(0, 0)\}) \times (\mathbb{R}^2 \setminus \{(0, 0)\}) \rightarrow (\mathbb{R}^2 \setminus \{(0, 0)\})$ by the following formula:

$$(a_1, b_1) \circ (a_2, b_2) = (a_1 a_2 - b_1 b_2, a_1 b_2 + a_2 b_1).$$

Show that $(\mathbb{R}^2 \setminus \{(0, 0)\}, \circ)$ is a group.

2. PROBLEM 2

(a) Let $\sigma, \tau \in S_5$ be the following permutations:

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 2 & 1 \end{bmatrix}, \quad \tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{bmatrix}.$$

Find the cycle decomposition of the permutation $\tau\sigma^2$.

(b) Find the order of the permutation $\tau\sigma^2$.

(c) Let $\phi \in S_{12}$ be the 12-cycle $(1\ 2\ 3 \dots 10\ 11\ 12)$. For which $i = 1, \dots, 11$, the power ϕ^i is also a 12-cycle?

3. PROBLEM 3

Let G be a group where all elements have order at most 2. Prove that G is abelian.

4. PROBLEM 4

(a) Let G be a group of order 40 and let H be a group of order 30. Let $\varphi: G \rightarrow H$ be a homomorphism. Prove that φ is *not* surjective.

(b) Construct a surjective homomorphism from group of order 40 to a group of order 20.

5. PROBLEM 5

(a) Prove or provide a counterexample: if A is a normal subgroup of B and B is a normal subgroup of C , then A is a normal subgroup of C .

(b) Prove that $\{0, 3\} \subset \mathbb{Z}/6\mathbb{Z}$ is a normal subgroup. Describe the quotient group $(\mathbb{Z}/6\mathbb{Z})/\{0, 3\}$.

6. PROBLEM 6

- (a) Prove that $\mathbb{Z} \subset \mathbb{Q}$ is a normal subgroup.
- (b) Consider the group $G := \mathbb{Q}/\mathbb{Z}$. Show that for any $g \in G$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $g^n = e_G$.
- (c) Let $H \subset \mathbb{C}^\times$ be the subset consisting of $h \in \mathbb{C}^\times$ such that $h^k = 1$ for some $k \in \mathbb{Z}_{\geq 0}$. Show that H is a subgroup.
- (d)* Show that the group H (defined in c)) is isomorphic to the group G (defined in b)).

7. PROBLEM 7

- (a) Let d be a divisor of n . Construct an injective homomorphism $D_{2d} \hookrightarrow D_{2n}$.
- (b) Prove that D_{2n} and $\mathbb{Z}/2n\mathbb{Z}$ are not isomorphic.
- (c) Find the center of the dihedral group D_{2n} .
- (d) Prove that D_{24} and S_4 are not isomorphic.