

① Symplectic duality

$X \leftarrow$ "symplectic singularity"

$\hookrightarrow$ (normal)
Poisson, affine, symplectic on X_{reg}

$\exists Y \xrightarrow{\pi} X$; $\pi^*(\omega_{\text{reg}}) \leftarrow$ extends to Y

$\hookrightarrow$
resolution

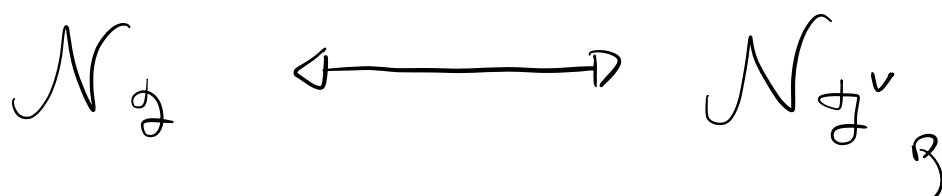
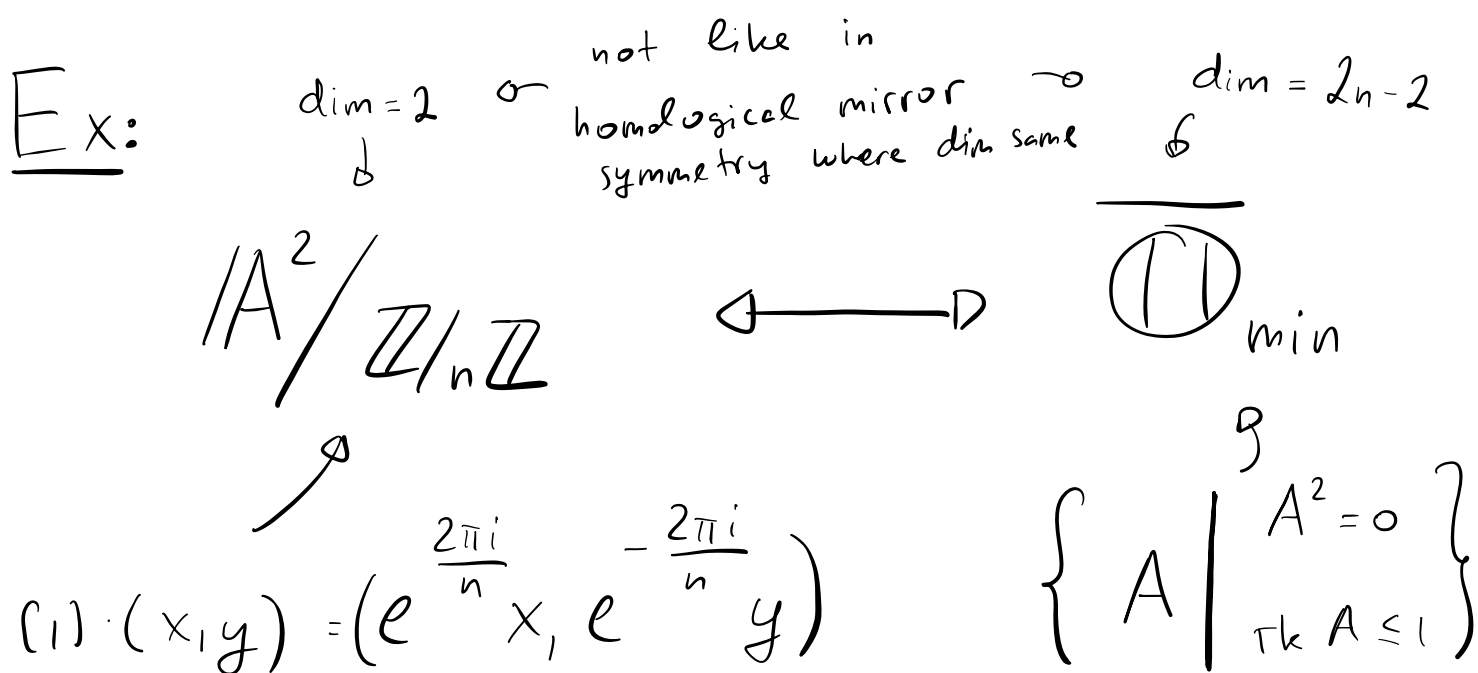
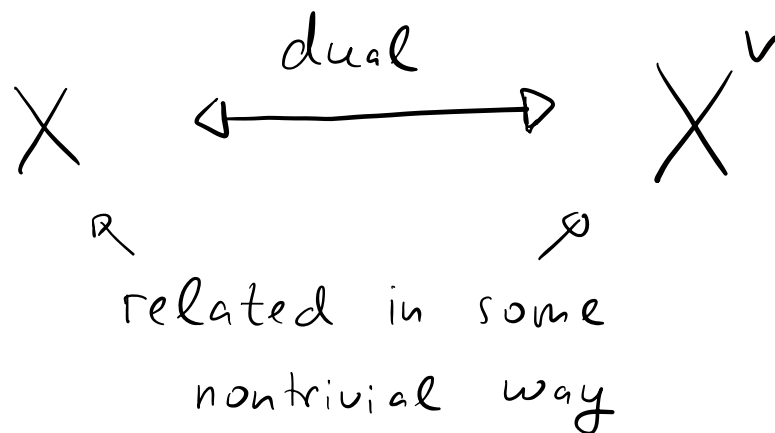
f.d. Lie alg.

E_X : $X = \mathcal{N}_{\mathfrak{g}} \leftarrow$ nilpotent cone in $\mathfrak{g}$

$X = \overline{\bigoplus \mathbb{C} \cdot e_i}$ $\leftarrow$ closure of nilp. orbit
(in type A)

$X = \mathbb{A}^2 / \Gamma = \text{Spec } \mathbb{C}[x, y]^{\Gamma}$, $\overset{\text{finite}}{\Gamma} \subset \text{SL}_2(\mathbb{C})$
 $\vdots$
 $\hookrightarrow$ "Kleinian singularity"

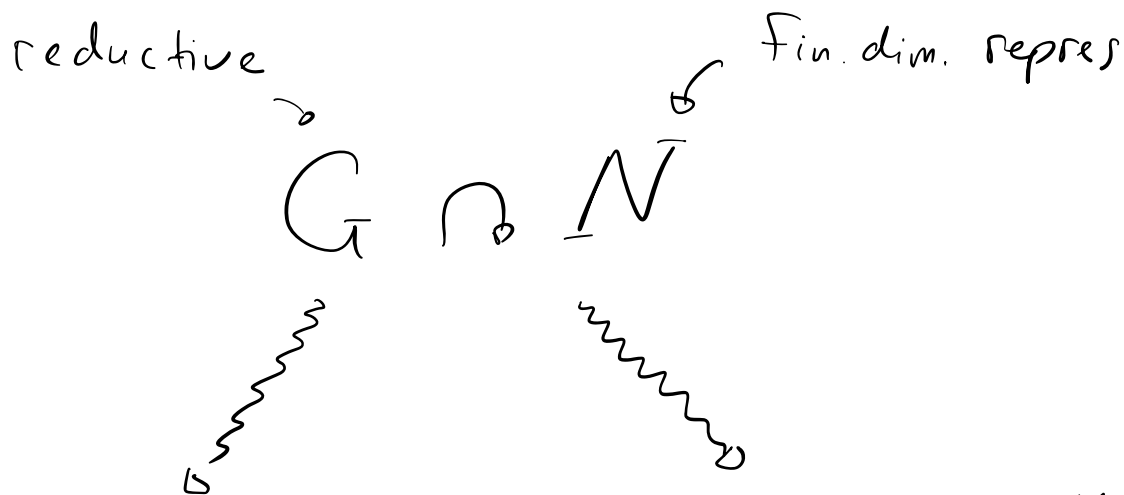
Symplectic duality:



Langlands dual

② Higgs and Coulomb branches

One "systematic" way to produce dual varieties $\leftrightarrow$ via "gauge theories"



$$X = \mathcal{M}_C$$

$\hookrightarrow$ "Coulomb branch"

$$\mathcal{M}_H = X^\vee$$

"Higgs branch"

Hamiltonian reduction

$$\mathcal{M}_H = T^*N // G$$

"BFN space"

$$\mathcal{M}_C = \text{Spec} \left(H_*^G \left(\mathcal{R}_{G,N}, \star \right) \right)$$

$\hookrightarrow$ convolution

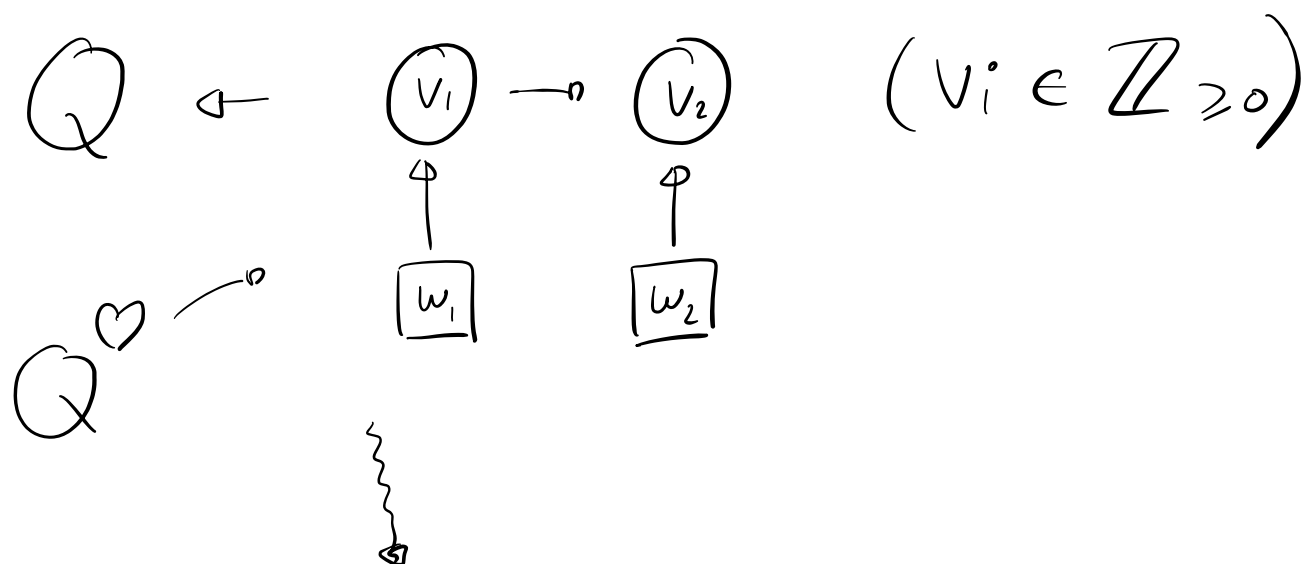
$\hookrightarrow$ Danya will tell us more

$$\mathbb{C}[\mu_c] \xrightarrow{\text{quantize}} \mathcal{A}_c = \left(H_*^{G \times \mathbb{C}^\times}(\mathcal{R}_{G,N}), \star \right)$$

$\xrightarrow{\text{assoc.}} \mathbb{C}[t], t=0 \sim \mathbb{C}[\mu_c]$

BFN's construction & works for $\forall G, N$,

distinguished class: "quiver gauge theories":

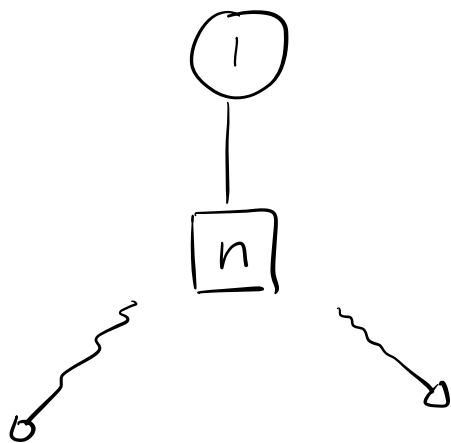


$$G = \prod GL_{v_i}; \quad N \leftarrow \begin{array}{c} \text{Hom}(\mathbb{C}^{w_1}, \mathbb{C}^{v_1}) \\ \oplus \\ \text{Hom}(\mathbb{C}^{v_1}, \mathbb{C}^{v_2}) \\ \oplus \\ \text{Hom}(\mathbb{C}^{w_2}, \mathbb{C}^{v_2}) \end{array}$$

$$\mathcal{M}_c(Q, \underline{v}, \underline{w}), \quad \mathcal{M}_H(Q, \underline{v}, \underline{w})$$

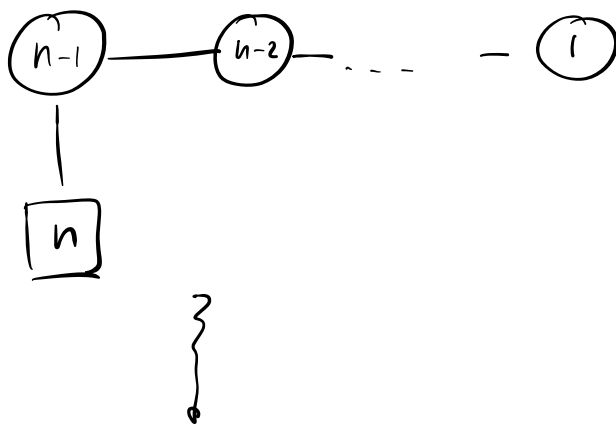
"Nakajima quiver var"

Examples:



$$\mathcal{M}_C = \mathbb{A}^2 / \mathbb{Z}_n \mathbb{Z}$$

$$\mathcal{M}_H = \overline{\mathbb{P}^1}_{\min}$$



$$\mathcal{M}_H \cong \mathcal{N}_{se_n} \cong \mathcal{M}_C$$

③ Coulomb branches for ADE Dynkin quiv.

$$Q \leftarrow \text{ADE} \quad \left(\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \bullet \begin{array}{l} \nearrow \bullet \\ \searrow \bullet \end{array} \end{array} \right)$$

$\leftarrow D_6$



$G_Q \leftarrow$ adjoint grp. corresp. to Q

$$\underline{w} \rightsquigarrow \lambda = \sum w_i \alpha_i \quad \begin{array}{l} \nearrow \text{fund. coweight} \\ \text{for } G_Q \end{array}$$

$$\underline{v} \rightsquigarrow \mu = \lambda - \sum v_i \alpha_i \quad \begin{array}{l} \nearrow \text{simple} \\ \text{coroots for } G_Q \end{array}$$

Thm [BFN] "generalized slice"
in $\text{Gr } G_Q$

$$\mathcal{M}_c(Q, \underline{v}, \underline{w}) \simeq W_\mu^\lambda$$

need a speaker!

$$W_H^\lambda = W_H \cap \overline{G_Q[z] \cdot z^\lambda \cdot G_Q[z]} \subset G_Q((z^{-1}))$$

$$T \cap$$

$$U \subset G_Q$$

$$U_-$$

φ

$$z^\lambda \in T((z^{-1}))$$

$$\begin{pmatrix} z^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & z^{\lambda_r} \end{pmatrix}^{\varphi}$$

$$W_H = \underbrace{U}_{\varphi \quad \overline{1}}[[z^{-1}]]_1 \cdot \underbrace{T}_{\varphi \quad \overline{2}}[[z^{-1}]]_1 \cdot z^H \cdot \underbrace{U}_{\overline{3}}[[z^{-1}]]_1$$

$\varphi \quad \overline{1}$

$\varphi \quad \overline{2}$

$\overline{3}$

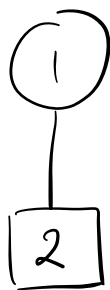
∞ -dim'l affine space!

iso to

product of $\overline{1}, \overline{2}, \overline{3}$

So, $\mathbb{C}[W_H^\lambda] \leftarrow$ quotient of $\mathbb{C}[W_H]$
by some ideal

Example / exercise:



$$\Rightarrow G_Q = \mathrm{PGL}_2$$

$$\lambda = 2\Theta_1$$

$$\mu = 0$$

$$W_0^{2\Theta_1} = \left\{ M = \begin{pmatrix} 1+a\tau^{-1} & b\tau^{-1} \\ c\tau^{-1} & 1-a\tau^{-1} \end{pmatrix} \mid \det M = 1 \right\}$$

$\parallel \quad (1+\tau^{-1} \cdot X)$

$\mathcal{N}_{\mathrm{se}_2} \ni \frac{1}{X}$

a^2+bc

More examples:

$$\mu \text{ is dominant} \Rightarrow W_\mu^\lambda \subset \mathrm{Gr}_{G_Q}$$

$\hookrightarrow$ "slice in affine Grassmannian"

[MV]

$$Q \text{ is type } A \Rightarrow W_\mu^\lambda \simeq \sum_{\varphi} e_\mu \cap \overline{\mathbb{D}}_{e_\lambda}$$

$\hookrightarrow$ "parabolic Slodowy Variety"

Kien will
tell us more

$$Se_\mu \cap \overline{\bigoplus} e_\lambda \longleftrightarrow Se_\lambda^\vee \cap \overline{\bigoplus} e_H^\vee$$

transposition
of Young diag.

Quantizations:

$$\mathbb{C}[W_\mu^\lambda] \cong H_{\bullet}^{G_\vee}(\mathcal{R})$$

quantities

truncated
shifted Yangian

$$Y_H^\lambda \cong H_{\bullet}^{G_\vee \times \mathbb{C}_t^\times}(\mathcal{R})$$

need a speaker!

$$\mathbb{C}[W_\mu] \longleftrightarrow \mathbb{C}[W_\mu^\lambda]$$

$$\langle E_i^{(k)}, H_i^{(r)}, F_i^{(k)} \rangle \longleftrightarrow Y_H^\lambda$$

" Y_H & shifted Yangian (additive version of

"gen. on" $U((\tau^{-1})_1)_1, T((\tau^{-1})_1)^\mathbb{H}, U_-((\tau^{-1})_1)_1$ $U_q(\hat{\mathfrak{g}}_{\mathbb{Q}})$)

Rem: $W_\mu^\lambda \leftarrow$ makes sense even if

$G_Q \leftarrow$ of types BCFG!

[NW] $\leftarrow$ generalize BEN's construction

to $Q \leftarrow$ symmetrizable and prove:

$$W_\mu^\lambda \simeq \mathcal{M}_C(Q, \underline{\nu}, \underline{\omega})$$

$\nearrow$ finite type Dynkin

$\mathcal{P}$

No $\mathcal{M}_H(Q, \underline{\nu}, \underline{\omega})$ in this case

What should be the dual thing?

One concrete conjecture that relates

dual sides so allows to extract information

about X from $X^\vee$ is:

④ Hikita conjecture

$$\mathcal{M}_c = W_H^\lambda \supset T \text{ or max. in } GQ$$

iso of graded algebras

Hikita:

$$\mathbb{C}[\mathcal{M}_c^T] \cong H^*(\tilde{\mathcal{M}}_H)$$

φ

resolution of $\mathcal{M}_H$

(Nakajima quiv.
variety)

schematic fixed points

$$(\mathbb{C}[\mathcal{M}_c] / (\mathbb{C}[\mathcal{M}_c]_\eta, \eta \neq 0))$$

$\eta: T \rightarrow \mathbb{C}^\times$

Example:

$$T = \mathbb{C}^\times$$

$\hookrightarrow$

$$\begin{aligned} \deg a &= 0 \\ \deg b &= 1 \\ \deg c &= -1 \end{aligned}$$

$$\mathbb{C}[\mathcal{M}_c] = \mathbb{C}[x, y] \stackrel{\mathbb{Q}/\mathbb{Z}}{=} \mathbb{C}[a, b, c] / a^n + bc$$

$$\mathbb{C}(u_c^T) = \mathbb{C}[a]/a^n = H^*(\underset{\substack{\text{resolution of } \overline{\mathbb{P}}_{n,n}}}{T \cdot \mathbb{P}^{n-1}})$$