

Character formulas of affine Lie algebras via the geometry of the Springer resolution

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$\mathfrak{g}$ - simple Lie algebra

$\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}d$ - affine Lie algebra

$$[at^n, bt^m] = [a, b]t^{n+m} + n(a, b)\delta_{m, -n}K$$

$\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$ - affine Cartan

Modules over $\widehat{\mathfrak{g}}$ and their characters

Pick $\Lambda \in \widehat{\mathfrak{h}}^*$, produce:

Verma module - $M(\Lambda) \twoheadrightarrow L(\Lambda)$ - the unique **simple** quotient

For $\widehat{\mathfrak{g}}$ -module M with finite dimensional $\widehat{\mathfrak{h}}$ -weight spaces, can consider:

$$\text{ch } M := \sum_{\mu \in \widehat{\mathfrak{h}}^*} \dim M_{\mu} e^{\mu} = \text{tr}_M e^h, \quad h \in \widehat{\mathfrak{h}}$$

$\text{ch } M(\Lambda)$ - easy

Question: can we compute $\text{ch } L(\Lambda)$?

Examples when the answer is known

Weyl-Kac formula:

$$\Lambda \text{ - integral, dominant: } \widehat{R} \operatorname{ch} L(\Lambda) = \sum_{w \in \widehat{W}} (-1)^{\ell(w)} e^{w(\Lambda + \widehat{\rho})}$$

Kac-Wakimoto formula [2017]

For $\mathfrak{g} = \mathfrak{sl}_n$, $\Lambda = -(1+s)\Lambda_0 + s\Lambda_1$ ($s \in \mathbb{Z}_{\geq 0}$) get:

$$\widehat{R} \operatorname{ch} L(\Lambda) = \sum_{u \in W} (-1)^{\ell(u)} u \sum_{\gamma \in Q^\vee, \langle \omega_1, \gamma \rangle \geq 0} e^{t_\gamma(\Lambda + \widehat{\rho})}$$

Question 1: what is so special about Λ 's in these formulas?

Question 2: can we generalize these formulas?

Assumptions on Λ

$$(\Lambda + \hat{\rho})(K) = \Lambda(K) + h^\vee > 0 \text{ \textcolor{red}{positive} level condition}$$

Λ - integral

Not crucial assumption:

Λ quasi-dominant meaning $\Lambda(\alpha_i^\vee) \geq 0$ for $i = 1, \dots, \dim \mathfrak{h}$

Pick $w_\nu \in \widehat{W}^f$ s.t. $w_\nu(\Lambda + \hat{\rho})$ - dominant and set $\lambda := w_\nu(\Lambda + \hat{\rho}) - \hat{\rho}$

$$\text{Kac-Kazhdan} \Rightarrow \widehat{R} \operatorname{ch} L(\Lambda) = \sum_{u \in W} (-1)^{\ell(u)} u \sum_{\gamma \in Q^\vee} a_\nu^\gamma e^{w_\gamma^{-1}(\lambda + \hat{\rho})}$$

Question: how to compute the numbers a_ν^γ ?

Kazhdan-Lusztig theory and aspherical module

$\mathcal{H}_q(\widehat{W})$ - affine Hecke algebra (deformation of $\mathbb{C}\widehat{W}$ over $\mathbb{C}[q^{\pm 1}]$)

$\mathcal{P} := \mathcal{H}_q(\widehat{W}) \otimes_{\mathcal{H}_q(W)} \mathbb{C}[q^{\pm 1}]$ - aspherical module

$\mathcal{P}$ has **standard** basis T_γ and **canonical** basis C_γ ($\gamma \in Q^\vee$)

Can decompose:

$$T_\gamma = \sum_{\nu \in Q^\vee} m_\nu^\gamma(q) C_\nu$$

Kashiwara-Tanisaki $\Rightarrow a_\nu^\gamma = m_\nu^\gamma(1)$

Geometric realization of the aspherical module

$\mathfrak{g}^\vee, G^\vee$ - Langlands dual Lie algebra and Lie group

Springer resolution:

$$\tilde{\mathcal{N}} := T^*(G^\vee/B^\vee) \xrightarrow{\pi} \mathcal{N} - \text{nilpotent cone}$$

Theorem [Kazhdan-Lusztig]:

There is an action $\widehat{W} \curvearrowright K^{G^\vee}(\tilde{\mathcal{N}})$; $t_\gamma \cdot - \mapsto [\mathcal{O}_{\tilde{\mathcal{N}}}(\gamma)] \otimes -$

Set $P := \mathcal{P}|_{q=1}$. We have an isomorphism of $\widehat{W}$ -modules:

$$P \simeq K^{G^\vee}(\tilde{\mathcal{N}}); \quad T_\gamma \mapsto [\mathcal{O}_{\tilde{\mathcal{N}}}(\gamma)]$$

Theorem [Bezrukavnikov]

There exists a **t-structure** in $D^b(\mathrm{Coh}^{G^\vee} \tilde{\mathcal{N}})$,

such that:

$$\text{irreducible objects} = \{E_\nu, \nu \in Q^\vee\}$$

$$K^{G^\vee}(\tilde{\mathcal{N}}) \ni [E_\nu] = C_\nu$$

So, both T_γ and C_ν are **classes** of some complexes of sheaves on $\tilde{\mathcal{N}}$

General approach to character formulas

So, want to find m_ν^γ , defined by:

$$[\mathcal{O}_{\tilde{\mathcal{N}}}(\gamma)] = \sum_{\nu \in Q^\vee} m_\nu^\gamma [E_\nu] \quad (1)$$

Pick

$$U \subset \mathcal{N} - G^\vee\text{-invariant}; \quad \tilde{U} \subset \tilde{\mathcal{N}} - \text{preimage of } U$$

Can restrict (1) to $\tilde{U}$:

$$[\mathcal{O}_{\tilde{U}}(\gamma)] = \sum_{\nu \in Q^\vee; \text{Supp } E_\nu \cap \tilde{U} \neq \emptyset} m_\nu^\gamma [E_\nu|_{\tilde{U}}]$$

Crucial observation:

$$\{[E_\nu|_{\tilde{U}}] =: \overline{C}_\nu, \text{Supp } E_\nu \cap \tilde{U} \neq \emptyset\} \text{ form a **basis** of } K^{G^\vee}(\tilde{U})$$

Upshot: $\text{ch } L(w_\nu^{-1} \circ \lambda)$ is controlled by $K^{G^\vee}(\tilde{U})$ if $\text{Supp } E_\nu \cap \tilde{U} \neq \emptyset$

Two-sided cells and examples of U

Theorem [Lusztig]

$$\{\text{nilpotent orbits in } \mathcal{N}\} \leftrightarrow \{\text{two-sided cells } c \subset \widehat{W}\}$$

$$\widehat{W} = \bigsqcup_{\mathbb{O} \subset \mathcal{N}} c_{\mathbb{O}},$$

$$\text{Supp } E_{\nu} \cap \widetilde{U} \neq \emptyset \text{ iff } \mathbb{O}_{w_{\nu}} \subset U$$

Example 0: take $U = \mathbb{O}_{\text{reg}}$, $c_{\text{reg}} = \{1\}$, $K^{G^{\vee}}(U) = \mathbb{C} \Rightarrow$ Kac-Weyl formula

Example 1: take $U = \mathbb{O}_{\text{reg}} \cup \mathbb{O}_{\text{subreg}}$, c_{subreg} - already interesting

$$c_{\text{subreg}} = \{w \in \widehat{W}, w \text{ has the } \mathbf{unique} \text{ minimal decomposition}\}$$

$$c_{\text{subreg}} \cap \widehat{W}^f = c_{\text{subreg}}^0 = \{w \in c_{\text{subreg}}, w = \dots s_0\}$$

Description of $K^{G^\vee}(\tilde{U})$

Have an exact sequence of $\widehat{W}$ -modules:

$$0 \rightarrow K^{G^\vee}(\tilde{\mathcal{O}}_{\text{subreg}}) \rightarrow K^{G^\vee}(\tilde{U}) \rightarrow \mathbb{C} \rightarrow 0$$

Define the graph Γ :

Vertices of Γ = elements of c_{subreg}^0

Vertices w and y are connected by an edge if they differ by simple reflection

$$W(\Gamma) \curvearrowright \mathfrak{h}_\Gamma; \quad \widehat{W} \hookrightarrow W(\Gamma) \quad s_j \mapsto \prod_{w=s_j \dots \in c_{\text{subreg}}^0} s_w$$

Theorem 1 [Bezrukavnikov-Kac-K.; Suzuki-K.]

a) We have the isomorphism of $\widehat{W}$ -modules:

$$K^{G^\vee}(\widetilde{\mathcal{O}}_{\text{subreg}}) \simeq \mathfrak{h}_\Gamma, \quad K^{G^\vee}(\widetilde{U}) \simeq \mathfrak{h}_\Gamma \oplus \mathbb{C}d$$

b) We can describe all complexes $E_\nu|_{\widetilde{U}}$ in $D^b(\text{Coh}^{G^\vee} \widetilde{U})$

c) Canonical basis consists of $\{\alpha_w, w \in c_{\text{subreg}}^0\} \cup \{d\}$

Theorem 2 [Bezrukavnikov-Kac-K.; Suzuki-K.]

Theorem 1 leads us to explicit character formulas for $\Lambda = w^{-1} \circ \lambda$ such that.

Case 1: $\lambda + \widehat{\rho}$ - regular, $w \in c_{\text{subreg}}^0$

Case 2: $\lambda = -\Lambda_i + \sum_j m_j \Lambda_j$ for $m_j \in \mathbb{Z}_{\geq 0}$ and $w = s_i \dots \in c_{\text{subreg}}^0$

Example (was conjectured by Kac-Wakimoto): $\mathfrak{g} = D_n$

$$\text{ch } L(-\Lambda_0) = \frac{1}{2} \sum_{u \in W} (-1)^{\ell(u)} u \sum_{\gamma \in Q^\vee} (\langle \theta, \gamma \rangle + 1) e^{t_\gamma(-\Lambda_0 + \widehat{\rho})}$$

Application

4D/2D correspondence:

$$\{4D \mathcal{N} = 2 \text{ SCFT}\} \rightarrow \{\text{vertex algebras}\}; \tau \mapsto V(\tau)$$

$$\text{ch } V(\tau) = \text{Schur index of } \tau$$

Our computation $\Rightarrow$ Schur indices of certain (nontrivial!) SCFT
such as Argures-Douglas theories

The corresponding $V(\tau)$ is $L_k(\mathfrak{g}) = L(k\Lambda_0)$ - simple quotient
of the universal vertex algebra $V^k(\mathfrak{g})$

Observation (via computations, should be a conceptual explanation):

for some $k \in \mathbb{Z}_{<0}$, characters of **all** ordinary irreducible modules
in the category $\mathcal{O}(L_k(\mathfrak{g}))$ are **covered** by our formulas