

Steinberg Varieties, Two-Sided Cells, and Categorifications of Affine Temperley-Lieb Algebras

Vasily Krylov

Harvard/CMSA

Based on joint works with Galina Dobrovolskaya, Ilya Dumanski,
Andrei Ionov, Ivan Karpov, and Elijah Bodish

Notations

G - simple Lie group over $\mathbb{C}$

$T \subset G$ - maximal torus, $\Lambda = \text{Hom}(\mathbb{C}^\times, T)$ - cocharacter lattice

W - Weyl group, $\widehat{W} = W \ltimes \Lambda$ - affine Weyl group

$\mathcal{H}_q(\widehat{W})$ - affine Hecke algebra (deformation of $\mathbb{C}\widehat{W}$ over $\mathbb{C}[q^{\pm 1}]$)

$\mathcal{P} := \text{Ind}_{\mathcal{H}_q(W)}^{\mathcal{H}_q(\widehat{W})} \mathbb{C}[q^{\pm 1}]$ - aspherical module over $\mathcal{H}_q(\widehat{W})$

$\mathbb{C}\widehat{W} = \mathcal{H}_q(\widehat{W})|_{q=1}$, $P := \mathcal{P}|_{q=1}$

Geometric realization of $\mathcal{H}_q(\widehat{W})$

$G^\vee$ - Langlands dual Lie group

Springer resolution and Steinberg variety:

$$\tilde{\mathcal{N}} := T^*(G^\vee/B^\vee) \xrightarrow{\pi} \mathcal{N} - \text{nilpotent cone}, \quad \text{St} := \tilde{\mathcal{N}} \times_{\mathcal{N}} \tilde{\mathcal{N}}$$

Theorem [Kazhdan-Lusztig]:

$$(K^{G^\vee \times \mathbb{C}^\times}(\text{St}), \star) \simeq \mathcal{H}_q(\widehat{W}) - \text{isomorphism of algebras}$$

$$K^{G^\vee \times \mathbb{C}^\times}(\tilde{\mathcal{N}}) \simeq \mathcal{P} - \text{isomorphism of modules}$$

Explicitly:

$$K^{G^\vee \times \mathbb{C}^\times}(\tilde{\mathcal{N}}) \simeq \mathcal{P} = (\mathbb{C}\Lambda)[q^{\pm 1}]; \quad [\mathcal{O}_{\tilde{\mathcal{N}}}(\gamma)] \mapsto \gamma$$

Canonical basis

$\mathcal{H}_q, \mathcal{P}$ – have canonical bases $C_w, \overline{C}_\gamma (w \in \widehat{W}, \gamma \in \Lambda)$

Relation between them:

$$\mathcal{H}_q \twoheadrightarrow \mathcal{P} \text{ - natural surjection, } C_w \mapsto \begin{cases} \overline{C}_{\gamma_w} & \text{if } w \text{ is shortest in } wW \\ 0 & \text{otherwise} \end{cases}$$

Why they are important:

$C_w|_{q=1} \in \mathbb{C}\widehat{W}$ control characters of irreducible $\widehat{\mathfrak{g}}$ -modules

$\overline{C}_\gamma|_{q=1} \in P$ - control $\mathfrak{g}$ -integrable irreducible $\widehat{\mathfrak{g}}$ -modules

Theorem [Bezrukavnikov]

There exists a t -structure in $D^b(\mathrm{Coh}^{G^\vee} \tilde{\mathcal{N}})$,

such that:

irreducible objects of its heart = $\{E_\gamma, \gamma \in Q^\vee\}$

$$K^{G^\vee}(\tilde{\mathcal{N}}) \ni [E_\gamma] = \overline{C}_\gamma$$

So, $\overline{C}_\gamma$ are classes of some complexes of sheaves on $\tilde{\mathcal{N}}$

Two-sided cells and filtrations

Theorem [Lusztig]

$$\widehat{W} = \bigsqcup_{\mathbb{O} \subset \mathcal{N}} c_{\mathbb{O}} \text{ - decomposition into two-sided cells}$$

$$\{\text{nilpotent orbits in } \mathcal{N}\} \leftrightarrow \{\text{two-sided cells } c \subset \widehat{W}\}$$

Theorem [Bezrukavnikov]

$$Z \subset \mathcal{N} \text{ closed } G^{\vee}\text{-invariant, } U := \mathcal{N} \setminus Z$$

$$\text{Supp } E_{\gamma} \subset \widetilde{Z} \text{ iff } \mathbb{O}_{w_{\gamma}} \subset Z$$

So,

$$K^{G^{\vee}}(\widetilde{Z}) = \text{Span}_{\mathbb{C}}(\overline{C}_{\gamma} \mid \mathbb{O}_{w_{\gamma}} \subset Z) \subset K^{G^{\vee}}(\widetilde{\mathcal{N}})$$

$$K^{G^{\vee}}(\widetilde{U}) = K^{G^{\vee}}(\widetilde{\mathcal{N}})/K^{G^{\vee}}(\widetilde{Z}) = \text{Span}_{\mathbb{C}}([\overline{C}_{\gamma}] \mid \mathbb{O}_{w_{\gamma}} \subset U)$$

Geometric way of thinking

Upshot:

$K^{G^\vee}(\tilde{\mathcal{N}})$ has a canonical basis that controls characters of $\hat{\mathfrak{g}}$ -modules

$$0 \rightarrow K^{G^\vee}(\tilde{Z}) \rightarrow K^{G^\vee}(\tilde{\mathcal{N}}) \rightarrow K^{G^\vee}(\tilde{U}) \rightarrow 0$$

s.e.s. of $K^{G^\vee}(\tilde{\mathcal{N}}) = \mathbb{C}\Lambda$ -modules compatible with canonical basis

Geometrically:

$\text{Spec } K^{G^\vee}(\tilde{\mathcal{N}}) = T^\vee$, canonical basis elements are **functions** on $T^\vee$

$T_U^\vee := \text{Spec } K^{G^\vee}(\tilde{U})$ - closed subvariety of $T^\vee$

$T_U^\vee \subset T^\vee$ zeroes of the ideal $I_Z := K^{G^\vee}(\tilde{Z})$

Description of $T_U^\vee$ and I_Z

Theorem [Ionov-Karpov-K.]

For $G = \mathrm{GL}_n$, we have

$$I_Z = (S_n \cdot \Pi_{\lambda^t} \mid \mathbb{O}_\lambda \subset Z), \quad T_U^\vee = \{t \in T^\vee \mid (w \cdot \Pi_{\lambda^t})(t) = 0\}$$

$$\Pi_{n_1, \dots, n_k} = \mathrm{Vand}_{n_1} \cdot \dots \cdot \mathrm{Vand}_{n_k} \in \mathbb{C}[x_i^{\pm 1}]$$

$$\Pi_{(3,2)} = (x_1 - x_2)(x_2 - x_3)(x_1 - x_3)(x_4 - x_5) \in \mathbb{C}[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1}, x_5^{\pm 1}]$$

Example:

$$U = \mathcal{N}_{\leq 2} = \bigcup_{(n-k,k)} \mathbb{O}_{(n-k,k)}$$

$$T_U = \{(x_1, \dots, x_n) \in (\mathbb{C}^\times)^n \mid (x_i - x_j)(x_j - x_k)(x_i - x_k) = 0\}$$

Idea of the proof

We deduce theorem above from the following result (conjectured by Roman Bezrukavnikov)

Proposition [Ionov-Karpov-K.]

For $G = \mathrm{GL}_n$, $I_Z \subset \mathbb{C}[T]$ is generated by isotypic components of irreducible S_n -modules $V(\lambda)$ such that $\mathbb{O}_\lambda \subset Z$

To prove this proposition, can pass to associated graded:

$$\mathrm{gr} \mathbb{C}[T] \simeq \bigoplus_{e \in \mathcal{N}/G^\vee} K^{Z_{G^\vee}(e)}(\mathcal{B}_e)$$

Then, using localization theorem in K -theory reduce to:

Proposition [Ionov-Karpov-K.]

For $G = \mathrm{GL}_n$, $e = e_\lambda \in \mathcal{N}$ and semisimple $s \in Z_{G^\vee}(e)$, the $\mathbb{C}[T^\vee]$ -module $K(\mathcal{B}_e^s)$ is generated by the isotypic component of $V(\lambda)$

Other types

It turns out that analogs of theorems above do **not** hold in general for other types

We say that (e, s) is good if $Z_{G^\vee}(C, s)^\circ$ is a Levi subgroup

Examples:

Any pair (e, s) is good for $G = \mathrm{GL}_n$

If s is such that $Z_{G^\vee}(s)^\circ$ is a Levi then (s, e) is good for any e

(s, e) is good if $s \in Z_{G^\vee}(e)^\circ$

Theorem [Ionov-Karpov-K.]

If (e, s) is good, then the $\mathbb{C}[T^\vee]$ -module $K(\mathcal{B}_e^s)$ is generated by the isotypic component of the Springer representation $H^{\mathrm{top}}(\mathcal{B}_e)$

Back to type A

$$U = \mathcal{N}_{\leq 2} = \bigcup_{(n-k,k)} \mathbb{O}_{(n-k,k)}$$

$$K^{\mathrm{GL}_n}(\widetilde{\mathcal{N}}_{\leq 2}) = \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] / ((x_i - x_j)(x_j - x_k)(x_i - x_k)) \quad - \widehat{W}\text{-module}$$

By direct computation can verify that the following elements act by zero:

$$(s_i - 1)(s_{i\pm 1} - 1)(s_i - 1) - (s_i - 1), \quad i \in \mathbb{Z}/n\mathbb{Z} \quad (1)$$

Setting $u_i = s_i - 1$ we see that (1) is the defining relation for $\mathrm{TL}_n^{\mathrm{aff}}$

$\mathrm{TL}_n^{\mathrm{aff}}$ - diagrams on a cylinder connecting n points on the top circle to n points on the bottom circle without intersections

Categorification of the affine Temperley-Lieb

We can prove the following result (mostly based on the literature!)

Set $\text{St}_{\leq 2} := \tilde{\mathcal{N}}_{\leq 2} \times_{\mathcal{N}} \tilde{\mathcal{N}}_{\leq 2}$

Theorem [Dobrovolskaya-Ionov-K.]

(a) There is an isomorphism of algebras:

$$(K^{\text{PGL}_n \times \mathbb{C}^\times}(\text{St}_{\leq 2}), \star) \simeq \text{TL}_n^{\text{aff}}$$

(b) Canonical basis in $K^{\text{PGL}_n \times \mathbb{C}^\times}(\text{St}_{\leq 2})$ is the basis of diagrams

(c) It categorifies to the basis of irreducible objects in the heart of a certain t -structure in $D^b(\text{Coh}^{G^\vee \times \mathbb{C}^\times} \text{St}_{\leq 2})$, relations (1) are satisfied at the categorical level

Relation to the affine Grassmannian

$G = \mathrm{GL}_n$, $\underline{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}^k$ - collection of fundamental weights

$\mathrm{Gr}_G^{\underline{\lambda}}$ - resolution of a Schubert variety in Gr_G

By connecting the geometry of $\mathrm{Gr}_G^{\underline{\lambda}}$ with the one of (parabolic) Springer resolutions we prove:

Theorem (in progress) [Dumanski-Bodish-K.]

- (a) There exists a t -structure in $D^b(\mathrm{Coh}^{G(\mathcal{O}) \rtimes \mathbb{C}^\times} \mathrm{Gr}_G^{\underline{\lambda}})$ coming from some tilting generator in this category, same is true for the analogs of Steinberg varieties in this setting
- (b) Classes of irreducible objects can be described algebraically as images of canonical basis elements in partial symmetrizations of aspherical modules for $\mathcal{H}^{\mathrm{aff}}$