PSET 5

Due date: 28/4 at 11.59 pm

Some problems are simple exercises, and some of them are more complicated. Please choose at least TWO of them and submit your solutions over email to:

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1. Lecture 10

1. Fill in the details of the argument in Lecture 10 claiming that $\operatorname{Spec}\mathbb{C}[\widetilde{\mathbb{O}}_{\mathfrak{p}}]$ is conical. Namely, recall that $\widetilde{\mathbb{O}}_{\mathfrak{p}} = G/H$ for some (possibly disconnected) $H \subset G$. Prove that:

(a) $\operatorname{Aut}_G(G/H) = N_G(H)/H$,

(b) pick $e \in \mathbb{O}_{\mathfrak{p}}$ and extend to an $\mathfrak{sl}_2$-triple e, h, f. Consider the corresponding coweight $t^h \colon \mathbb{C}^\times \to G$. Check that t^h normalizes both $Z_G(e)$ and $H \subset Z_G(e)$,

(c) using (b) we obtain the action of $\mathbb{C}^\times$ on $\widetilde{\mathbb{O}}_{\mathfrak{p}}$ and so also on $\operatorname{Spec}\mathbb{C}[\widetilde{\mathbb{O}}_{\mathfrak{p}}]$, prove that the latter is conical (use the finite extension $\mathbb{C}[\overline{\mathbb{O}}_{\mathfrak{p}}] \hookrightarrow \mathbb{C}[\widetilde{\mathbb{O}}_{\mathfrak{p}}]$).

2. Lecture 11

2. Consider the parabolic Slodowy variety $\widetilde{S}(\mathfrak{a}, \mathfrak{p})$ and its deformation

$$\varphi \colon \widetilde{S}_{\mathfrak{h}}(\mathfrak{a}, \mathfrak{p}) \to \mathfrak{h}$$

over the space $\mathfrak{h} = (\mathfrak{p}/[\mathfrak{p}, \mathfrak{p}])^*$ that can be identified with $\mathfrak{z}(\mathfrak{m})$. Prove that the fiber of φ over generic semisimple element $s \in \mathfrak{z}(\mathfrak{m})$ is isomorphic to

$$S_a \cap \mathbb{O}_s.$$

3. Lecture 12

3. Consider the homogeneous universal enveloping algebra $\mathcal{U}_\hbar(\mathfrak{sl}_2)$. Prove that the element $C := 2ef + 2fe + h^2 \in \mathcal{U}_\hbar(\mathfrak{sl}_2)$ is central.

4. Recall that

$$\mathcal{A}_{\hbar,c} := \mathcal{U}_\hbar(\mathfrak{sl}_2)/(C - c(c+2)\hbar^2),\ \ D_{\hbar,c}(\mathbb{P}^1) = \left(D_\hbar(\mathbb{A}^2)/x\partial_x + y\partial_y - c\hbar\right)^{\mathbb{C}^\times}$$

Prove that the map

$$e \mapsto -y\partial_x,\ \ f \mapsto -x\partial_y,\ \ h \mapsto y\partial_y - x\partial_x$$

induces the isomorphism $\mathcal{A}_{\hbar,c} \simeq D_{\hbar,c}(\mathbb{P}^1)$ (you can use that $\operatorname{gr} D_{\hbar,c}(\mathbb{P}^1) = \mathbb{C}[T^*\mathbb{P}^1]$).

4. Lecture 13

5. Prove that a finitely generated module M over $\mathfrak{g}$ lies in the category $\mathcal{O}$ iff the element h (see Lecture 13) acts on M with finite dimensional generalized eigenspaces and real parts of eigenvalues of $h \curvearrowright M$ are bounded from the above.

6. For $Y = T^*\mathcal{B}$ prove that cocharacter $\nu\colon \mathbb{C}^\times \to T$ is generic (i.e., $Y^{\nu(\mathbb{C}^\times)} = Y^T$) iff $\langle \nu, \alpha\rangle \neq 0$ for any root α.

7. Let ν be a generic cocharacter of T. Prove that the Cartan subquotient $\mathsf{C}_\nu(\mathcal{U}(\mathfrak{g}))$ is isomorphic to $\mathcal{U}(\mathfrak{h})$ (see Lecture 13 for the definition of C_ν as well as for the idea for the argument).

8. Again see Lecture 13 for the details. Check that we natural map $\mathsf{C}_\nu(A) \to \operatorname{gr} \mathsf{C}_\nu(\mathcal{A})$ is surjective.

5. Lecture 14

9. Let $\Delta_j \in \mathcal{A}$-mod be a Verma module (see Lecture 14). Show that the generalized eigenspaces of $h \curvearrowright \Delta_j$ are frinite-dimensional (use that $\mathcal{A}_i$ is finitely generated over $\mathcal{A}_0$).

6. Lecture 15

10. Conside the Poisson (commutative) algebra $A = \mathbb{C}[[x, y]]$ with bracket given by $\{x, y\} = 1$. Prove that any formal quantization of A (i.e., the quantization over $\mathbb{C}[[\hbar]]$) is isomorphic to the Weyl algebra $\widehat{W}_\hbar = \mathbb{C}\langle\langle x, \partial_x, \hbar\rangle\rangle/([\partial_x, x] = \hbar)$.

11. Consider the Weyl algebra

$$W = \mathbb{C}\langle x, \partial_x\rangle/([\partial_x, x] = 1) = D(\mathbb{A}^1).$$

It is graded by $\deg x = 1$, $\deg \partial_x = -1$. Prove that the Cartan subquotient w.r.t. this grading is (canonically) isomorphic to $\mathbb{C}$.

12. Consider the universal quantization $\mathcal{A}_{\mathfrak{h}}$ and consider the grading induced some cocharacter $\nu\colon \mathbb{C}^\times \to T_X$. Fix $\lambda \in \mathfrak{h}$, prove that

$$\mathsf{C}_\nu(\mathcal{A}_\lambda) = \mathsf{C}_\nu(\mathcal{A}_{\mathfrak{h}})_\lambda.$$

7. Lecture 17

13. At the level of $\mathbb{C}$-points check that $\mathrm{Gr}_G^T = \mathrm{Gr}_T$.

8. Lecture 19

14. At the level of $\mathbb{C}$-ponts check that $G[z]z^\lambda G[z]$ consists of $g \in G((z^{-1}))$ such that $\Delta^{(s)}_{\beta,v}(g) = 0$ for $s > \langle \lambda, -w_0(\eta^\vee)\rangle$, where $\eta^\vee$ run through all dominant weights of G, $v \in V_{\eta^\vee}$, $\beta \in V^*_{\eta^\vee}$ and $\Delta^{(s)}_{\beta,v}(g)$ is the coefficient in front of z^{-s} of the pairing $\langle g\beta, v\rangle$.

9. Lecture 20

15. Prove that $\widetilde{\mathrm{Gr}}{}_G^{\underline{\lambda}}$ is smooth if all λ_i are minuscule.

16. Let U be a unipotent aalgebraic group. Show that the natural morphism $U[[z^{-1}]]_1 \times U[z] \to U((z^{-1}))$ is an isomorphism.