# PSET 4

Due date: 30/3 at 11.59 pm

Some problems are simple exercises, and some of them are more complicated. Please choose at least ONE of them and submit your solutions over email to:

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**1.** Let $\widetilde{\mathbb{O}}$ be a finite cover of a nilpotent orbit $\mathbb{O}$. In Lecture 10 we defined the $\mathbb{C}^\times$-action on $\mathbb{C}[\widetilde{\mathbb{O}}]$ and proved that $\mathbb{C}[\widetilde{\mathbb{O}}]_i = 0$ for $i < 0$. Prove that $\mathbb{C}[\widetilde{\mathbb{O}}]_0 = \mathbb{C}$. This would imply that the action $\mathbb{C}^\times \curvearrowright \operatorname{Spec}\mathbb{C}[\widetilde{\mathbb{O}}]$ is conical.

**2.** Let $P$ be parabolic with Levi $M \subset P$. Set $\mathfrak{h} := (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^*$ and consider the (universal) deformation $\widetilde{\varphi}\colon T^*_{\mathfrak{h}}(G/P) := G \times^P [\mathfrak{p},\mathfrak{p}]^\perp \to \mathfrak{h}$ (see Lecture 10). Identify $\mathfrak{h} \simeq \mathfrak{z}(\mathfrak{m})$. Pick generic $s \in \mathfrak{z}(\mathfrak{m})$, describe the fiber $\widetilde{\varphi}^{-1}(s)$. Can you describe this fiber for arbitrary $s \in \mathfrak{z}(\mathfrak{m})$?

**3.** Construct canonical identification $H^2(T^*(G/P),\mathbb{C}) \simeq (\mathfrak{p}/[\mathfrak{p},\mathfrak{p}])^*$.

**4.** Let $e$ be the subregular nilpotent element in $\mathfrak{sl}_n$ and consider the Slodowy variety $\widetilde{S}_e \subset \widetilde{\mathcal{N}}$. Let $\mathbb{O} \subset \mathfrak{sl}_n$ be the subset of rank one matrices and let $\overline{\mathbb{O}} = \mathbb{O} \cup \{0\}$ be its closure. Let $T \subset \mathrm{SL}_n$ be the subgroup of diagonal matrices. Consider the conjugation action $T \curvearrowright \overline{\mathbb{O}}$ and let $\overline{\mathbb{O}}^T$ be the *schematic* fixed points under this action (equivalently, $\overline{\mathbb{O}}^T$ is the schematic intersection of $\overline{\mathbb{O}}$ with $\mathfrak{t}$ inside $\mathfrak{sl}_n$). Check that there exists an isomorphism of algebras:

$$H^*(\widetilde{S}_e) \simeq \mathbb{C}[\overline{\mathbb{O}}^T].$$

**Problem for the short paper.** Here I propose one problem for a short paper that registered undergraduate students should write to complete the course.

In one of the lectures I used that the closure of arbitrary nilpotent orbit in type $A$ is normal. The goal of this project is to prove this fact via Frobenius splitting. See [MV] for the details.

## References

[MV] Mehta, V.B. and Van der Kallen, W., 1992. A simultaneous Frobenius splitting for closures of conjugacy classes of nilpotent matrices. Compositio Mathematica, 84(2), pp.211-221.