

PSET 3

DUE DATE: 7/3 AT 11.59 PM

Some problems are simple exercises, and some of them are more complicated. Please choose at least TWO of them and submit your solutions over email to:

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1. Let $X = \text{Spec } A$ be an affine variety. Assume that we are given a conical action $\mathbb{C}^\times \curvearrowright X$. Let $\mathfrak{m} \subset A$ be the maximal ideal of the (unique) \mathbb{C}^\times -fixed point on X . We have a grading $A = \bigoplus_{i \geq 0} A_i$ that induces the grading on $A^{\wedge m}$ (completion of A at \mathfrak{m}). Prove that

$$A = \bigoplus_{i \geq 0} (A^{\wedge m})_i.$$

2. This is an exercise from Lecture 5 so we follow notations used in this lecture. Recall that every nilpotent orbit \mathbb{O} is a symplectic subvariety of \mathfrak{g} . Moreover, we proved that the intersection $S_e \cap \mathbb{O}$ is also symplectic with the symplectic form being a restriction from the one on \mathbb{O} . We can also form the Hamiltonian reduction $\mathbb{O} //_{\chi} Q$ and it follows from Lecture 5 that we have the isomorphism

$$(S_e \cap \mathbb{O}) \times Q \simeq \mu_{\mathbb{O}}^{-1}(\chi)$$

inducing the isomorphism

$$S_e \cap \mathbb{O} \simeq \mathbb{O} //_{\chi} Q.$$

Check that this isomorphism is compatible with symplectic structures on these varieties.

3. Check that the subregular nilpotent orbit $\mathbb{O}_{\text{subreg}}$ in type A contains any other nonregular orbit in its closure.

4. Let $\Gamma \subset \text{SL}_2$ be a finite subgroup corresponding to \mathfrak{g} of type D or E (via the McKay correspondence). Feel free to assume that $\mathfrak{g} = \mathfrak{so}_8$ and $\Gamma = Q_8$ the quaternion group consisting of $\{\pm 1, \pm i, \pm j, \pm k\}$ acting on $\mathbb{C}^2 = \mathbb{H}$ via the left multiplication.

Consider the corresponding Kleinian singularity $X = \mathbb{A}^2/\Gamma = \text{Spec}(\mathbb{C}[x, y]^\Gamma)$.

Check that $\text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X)$ is finite (can you describe it for $\mathfrak{g} = \mathfrak{so}_8$, it will have to do with the automorphisms of the diagram D_4 ?).

Possible steps for the proof are:

1) Consider $X^{\text{reg}} := (\mathbb{A}^2 \setminus 0)/\Gamma$, this is smooth and symplectic (you can use that the action of Γ on $\mathbb{A}^2 \setminus 0$ is free). Using that X is normal deduce that

$$\text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X) = \text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X^{\text{reg}}).$$

2) We proved that $\text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X)$ is an algebraic group so to see that $\text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X)$ is finite it is enough to show that $\text{Lie } \text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X^{\text{reg}}) = \{0\}$.

3) Consider the étale cover $\mathbb{A}^2 \setminus 0 \rightarrow X^{\text{reg}}$, show that it induces the embedding

$$\text{Lie } \text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(X^{\text{reg}}) = \{0\} \hookrightarrow \text{Lie}(\text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(\mathbb{A}^2 \setminus 0))^\Gamma.$$

4) Prove that $\text{Aut}_{\{\cdot, \cdot\}}^{\mathbb{C}^\times}(\mathbb{A}^2 \setminus 0) = \text{SL}_2$. It remains to check that $\mathfrak{sl}_2^\Gamma = \text{Lie } Z_{\text{SL}_2}(\Gamma) = 0$.

5) Pick any element $s \in \Gamma$ such that $s \neq \pm 1$. Show that $Z_{\text{SL}_2}(s) = \mathbb{C}^\times$ (use that s is semisimple). Deduce that $\text{Lie } Z_{\text{SL}_2}(\Gamma)$ is nonzero only if Γ is conjugated to a subgroup $\{\text{diag}(t, t^{-1}) \mid t \in \mathbb{C}^\times\} \subset \text{SL}_2$.

6) Finish the argument by showing that any finite subgroup of $\{\text{diag}(t, t^{-1}) \mid t \in \mathbb{C}^\times\}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Problem for the short paper. Here I propose one problem for a short paper that registered undergraduate students should write to complete the course.

Work out the argument of Slodowy (see [S, Section III]) proving that S_e is isomorphic to \mathbb{A}^2/Γ for e being subregular in \mathfrak{g} and $\Gamma \subset \text{SL}_2$ being the finite subgroup corresponding to \mathfrak{g} (via the McKay correspondence).

REFERENCES

- [S] Slodowy, Peter. *Simple singularities and simple algebraic groups* (2006): 70-102.