PSET 2

DUE DATE: 16/2 AT 11.59 PM

Some problems are simple exercises, and some of them are more complicated. Please choose at least TWO of them and submit your solutions over email to:

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1. Let $\mathfrak{g}$ be any Lie algebra. Pick an adjoint G-orbit $\mathbb{O} \subset \mathfrak{g}^*$ and define the form $\omega \in \Gamma(\mathbb{O}, \Lambda^2 T^*_{\mathbb{O}})$ as we did in Lecture 3. Check that $d\omega = 0$.

2. Let G be a group acting on an smooth algebraic variety X. Check that the induced action $G \curvearrowright T^*X$ is Hamiltonian (we consider the standard symplectic structure on T^*X) with the moment map $\mu\colon T^*X \to \mathfrak{g}^*$ given by

$$\mu^*(x) = \xi_x \in \mathrm{Vect}_X \subset \mathbb{C}[T^*X] \text{ for } x \in \mathfrak{g},$$

where ξ_x is the vector field on X defining the infinitesimal action of $x \in \mathfrak{g}$ on X.

3. Let G be a group and let $P \subset G$ be its subgroup. Recall the identification $T^*(G/P) \simeq G \times^P \mathfrak{p}^\perp$ discussed in Lectures 3,4. Show that the moment map from Exercise 3 is given by the formula

$$T^*(G/P) = G \times^P \mathfrak{p}^\perp \ni [g,p] \mapsto \mathrm{Ad}_g(p).$$

4. Using Problem 3 prove that $T^*(G/P)$ is isomorphic to the Hamiltonian reduction of T^*G by the (right) action of P.

5. (a) Consider the type A_1 Kleinian singularity $X = \mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z}) = \mathbb{C}[x^2, y^2, xy]$. Prove that the blow up $\pi\colon \mathrm{Bl}_{(0,0)} X \to X$ of X at the point $(0,0)$ is isomorphic to $T^*\mathbb{P}^1$. Prove that the pull back of the symplectic form on $X^{\mathrm{reg}} := (X \setminus \{(0,0)\})/(\mathbb{Z}/2\mathbb{Z})$ to $\pi^{-1}(X^{\mathrm{reg}}) \subset \mathrm{Bl}_{(0,0)} X$ extends to the symplectic form on $\mathrm{Bl}_{(0,0)} X$.

(b)* Let's now consider $X = \mathbb{A}^2/\mathbb{Z}/n\mathbb{Z}$, prove that it has a resolution obtained by making $n-1$ blow ups at appropriate points. Check that the pull back of the symplectic form on $X^{\mathrm{reg}} := (\mathbb{A}^2 \setminus \{(0,0)\})/(\mathbb{Z}/n\mathbb{Z})$ to the blow up as above extends to the symplectic form (an alternative way of seeing this is via the identification of $\mathbb{A}^2/(\mathbb{Z}/n\mathbb{Z})$ with S_e for subregular nilpotent e).

6. Consider the embedding

$$G \times S_e^{\mathfrak{g}} \subset G \times \mathfrak{g} \simeq G \times \mathfrak{g}^*.$$

Prove that the restriction of the symplectic form on $G \times \mathfrak{g}^* \simeq T^*G$ defines a symplectic form on $G \times S_e^{\mathfrak{g}}$ (you may consult with [L, Section 2]).

Problem for the short paper. Here I propose one problem for a short paper that registered undergraduate students should write to complete the course.

I propose to read, understand and then summarize the short paper [L] by Ivan Losev. It proves an important result describing local structure of arbitrary smooth affine Poisson G-varieties and constructing the analogs of slices we studied in this more general setting.

References

[L] Losev, I. V. (2006). *Symplectic slices for actions of reductive groups*. Sbornik: Mathematics, 197(2), 213.