PSET 1

DUE DATE: 9/2 AT 11.59 PM

Some problems are simple exercises, and some of them are more complicated. Please choose at least three of them and submit your solutions over email to:

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1. Let X be an affine algebraic variety over complex numbers equipped with an action of $\mathbb{C}^\times$. Set $A := \mathbb{C}[X]$. Consider the action $\mathbb{C}^\times \curvearrowright A$ given by the formula $t.f(x) := f(tx)$. It induces the $\mathbb{Z}$-grading $A = \bigoplus_{i\in\mathbb{Z}} A_i$:

$$A_i := \{f \in A \,|\, t.f = t^i f\}.$$

Recall that X is called conical if for every $x \in X$, the limit $\lim\limits_{t\to 0} t.x$ exists and is equal to the unique $\mathbb{C}^\times$-fixed point on X.

Prove that X is conical iff $A_i = 0$ for $i < 0$ and $A_0 = \mathbb{C}$.

2. Let ω be a symplectic form on some smooth affine variety $X = \operatorname{Spec} A$. Assume also that X is equipped with a $\mathbb{C}^\times$-action such that ω is homogeneous of negative degree $-d$ (i.e., $t^*\omega = t^{-d}\omega$ for any $t \in \mathbb{C}^\times$). Let $\{\,,\,\}\colon A \times A \to A$ be the Poisson bracket induced by ω (see Lecture 1). Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be the $\mathbb{Z}$-grading induced by the $\mathbb{C}^\times$-action. Check that $\{A_i, A_j\} \subset A_{i+j-d}$.

3. Let $\pi\colon Y \to X$ be a symplectic resolution. Check that the induced homomorphism $\pi^*\colon \mathbb{C}[X] \to \mathbb{C}[Y]$ is an isomorphism.

4. Recall the variety:

$$X_k^n := \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \,|\, A, B, C, D \in \mathbb{C}[z],\ D = z^k + \ldots,\ \deg B, \deg C < k,\ \det M = z^n \right\}.$$

Consider the $\mathbb{C}^\times$-action on X_k^n given by the formula

$$\begin{pmatrix} A(z) & B(z) \\ C(z) & D(z) \end{pmatrix} \mapsto \begin{pmatrix} t^{n-k}A(t^{-1}z) & t^k B(t^{-1}z) \\ t^{n-k}C(t^{-1}z) & t^k D(t^{-1}z). \end{pmatrix}$$

Check that this $\mathbb{C}^\times$-action is conical iff $n \geqslant 2k$.

5. The goal of this exercise is to prove the Jacobson-Morozov theorem. Recall that $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ and G is a connected Lie group with $\operatorname{Lie} G = \mathfrak{g}$.

Theorem 1. Every nilpotent element $e \in \mathfrak{g}$ can be included into an $\mathfrak{sl}_2$-triple: there exist elements $h, f \in \mathfrak{g}$ such that $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$.

The following two results also hold (see, for example, [CG, Sectioon 3.7]).

Theorem 2. Let (e, h, f) and (e, h', f') be two $\mathfrak{sl}_2$-triples. Then there is an element $g \in G$ centralizing e such that $gh = h'$, $gf = f'$.

Theorem 3. Let (e, h, f) and (e', h, f') be two $\mathfrak{sl}_2$-triples. Then there is an element $g \in G$ centralizing h such that $ge = e'$, $gf = f'$.

(1) Prove Theorem 1 in the case of $\mathfrak{g} = \mathfrak{sl}_n$.
(2) Prove Theorem 1 for general $\mathfrak{g}$. You may use the following strategy:
   (a) Check that $x \in \mathfrak{g}$ lies in the image of $\operatorname{ad} e$ if and only if x is orthogonal (with respect to the Killing form) to the centralizer of e.
   (b) Prove Theorem 1 in the case when the centralizer of e consists of nilpotent elements.
   (c) Prove Theorem 1 in the general case.

   In the following problems feel free to use Theorems 1 and 2 above.
(3) Show that the nilpotent orbits in $\mathfrak{g}$ are in one-to-one correspondence with the G-conjugacy classes of Lie algebra homomorphisms $\mathfrak{sl}_2 \to \mathfrak{g}$.
(4) Show that the number of nilpotent orbits in $\mathfrak{g}$ is finite.

6. Let $\mathfrak{g}$ be a simple Lie algebra (over $\mathbb{C}$). Let $\mathfrak{b} \subset \mathfrak{g}$ be a Borel subalgbra. Consider the Killing form $(\, , \,)$ on $\mathfrak{g}$ given by the formula $(x, y) := \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_y)$. It is known to be nondegenerate so it defines the identification $\mathfrak{g}^* \simeq \mathfrak{g}$. Let $\mathfrak{n} \subset \mathfrak{b}$ be the nilpotent radical. Check that the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ gives the identification $\mathfrak{b}^\perp \simeq \mathfrak{n}$.

7. Let $\mathbb{O} \subset \mathfrak{g}$ be a nilpotent orbit. Consider the $\mathbb{C}^\times$-action on $\mathfrak{g}$ given by $x \mapsto tx$. Check that $\mathbb{O}$ is $\mathbb{C}^\times$-stable.

8. Let $\mathfrak{g} = \mathfrak{sl}_{r+1}$, $G = \mathrm{SL}_{r+1}$ and consider the SL_{r+1}-invariant function on $\mathfrak{g}$:

$$\Phi_i := (A \mapsto \operatorname{tr}(A^{i+1})), \; i = 1, \ldots, r.$$

Check that the differentials of Φ_i are linearly independent at $A \in \mathfrak{g}$ iff A is regular (i.e., the centralizer of A in $\mathfrak{g}$ has dimension r).

9. (a) Let X be an affine irreducible Cohen-Macaulay variety (that is, $A = \mathbb{C}[X]$ is a Cohen-Macaulay ring). Prove that there exists a finite morphism $f \colon X \to V$ to the affine space $V = \mathbb{A}^{\dim X}$ such that $f_* \mathcal{O}_X$ is locally free (hint: use Noether normalization and miracle flatness, any finite morphism as above would work).

(b) Use (a) to prove that an affine Cohen-Macaulay variety that has an nonempty open reduced subvariety must be reduced.

(c) Using Problem 9 deduce that the nilpotent cone (defined as $\operatorname{Spec}\mathbb{C}[\mathfrak{g}]/(\Phi_i)$) is reduced in type A, this is the part that we did not discuss in the second lecture.

10. Write the detailed argument deducing that the morphism $G \times S_e^{\mathfrak{g}} \to \mathfrak{g}$ is smooth from the transverality of $\mathbb{O}_e$ and $S_e^{\mathfrak{g}}$ at the point $\{e\} = \mathbb{O}_e \cap S_e^{\mathfrak{g}}$ (see Lecture 2).

Problem for the short paper. Here we propose one problem for a short paper that registered undergraduate students should write to complete the course.

Consider the variety X_k^0. Recall that it can be identified with the moduli space of maps

$$\Big\{ f\colon \mathbb{P}^1 \to \mathbb{P}^1 \,|\, f(\infty) = \infty,\, \deg f = k \Big\}.$$

The goal of this project is to define the symplectic form in this space and then to prove that in the local étale coordinates x^i, y^i (see Lecture 1) this form is given by the formula $\omega = \sum_i \frac{dx^i \wedge dy^i}{y^i}$. Useful references are [AH] and [FKMM].

References

[AH] M. F. Atiyah, N. Hitchin, *The Geometry and Dynamics of Magnetic Monopoles*, Princeton University Press, (1988).

[CG] N. Chriss, V. Ginzburg, *Introduction to the geometric representation theory*, Birkhäuser Boston, Inc., Boston, MA, (1997).

[FKMM] M. Finkelberg, A. Kuznetsov, N. Markarian, and I. Mirković, *A Note on the Symplectic Structure on the Space of G-Monopoles*, Communications in Mathematical Physics, vol. 201, no. 3, pp. 411–421, 1999.