Practice Final Exam

Course: *Math 122*

Date: December 4, 2025

Duration: **180 minutes**

Name: ______________________________

Student ID: ______________________________

Instructions:

- Each problem is worth **10 points**. The total number of points you can earn on this exam is **100**.
- You are allowed to bring **one single-sided reference sheet** on **8.5×11 inch letter paper** (handwritten or typed).
- The duration of the exam is **180 minutes**.
- Write all solutions clearly in the space provided. Show all work for full credit.
- No additional materials or electronic devices are allowed.

1. (a) (5 points) Prove that

$$\det\colon \mathrm{Mat}_{2\times 2}(\mathbb{C}) \to \mathbb{C}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$$

is a homomorphism of monoids $(\mathrm{Mat}_{2\times 2}(\mathbb{C}), \cdot)$, $(\mathbb{C}, \cdot)$.

(b) (5 points) Consider matrices:

$$A = \begin{pmatrix} 5 & 10 \\ 2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & 11 \\ 6 & 3 \end{pmatrix}.$$

Compute the product AB. Is it invertible?

Hint: use that $\det\colon \mathrm{Mat}_{2\times 2}(\mathbb{C}) \to \mathbb{C}$ is a homomorphism of monoids.

2. Let $a, b, c \in S_5$ be the following permutations:

$$a = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & 5 & 2 \end{bmatrix}, \ b = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & 5 & 3 & 2 \end{bmatrix}, \ c = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 2 & 1 \end{bmatrix}.$$

(a) (5 points) Find the cycle decomposition of the permutation ab^2c.

(b) (5 points) Find the order of the permutation ab^2c.

3. (10 points) Consider the element $(1\,2\,3\,4) \in S_6$. Recall that the centralizer of $(1\,2\,3\,4)$ is the following subgroup of S_6:

$$G := \{\sigma \in S_6 \,|\, \sigma(1\,2\,3\,4)\sigma^{-1} = (1\,2\,3\,4)\} \subset S_6.$$

Find the order of G.

4. (10 points) Let G be a group and assume that $\varphi\colon G \twoheadrightarrow \mathbb{Z}/k\mathbb{Z}$ is a surjective homomorphism with kernel isomorphic to $\mathbb{Z}/3\mathbb{Z}$. Assume also that there exists an element $g \in G$ of order k such that $\varphi(g) = 1$.

   Show that G is isomorphic to either $(\mathbb{Z}/k\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ or $(\mathbb{Z}/k\mathbb{Z}) \ltimes (\mathbb{Z}/3\mathbb{Z})$, where the semidirect product is induced by the homomorpism:

   $$\mathbb{Z}/k\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/3\mathbb{Z})$$

   given by $1 \mapsto (x \mapsto -x)$ (note that this homomorphism exists only if k is even).

   Hint: consider the subgroup of G generated by the element $g \in G$.

5. (10 points) Explicitly describe groups of order 14 up to isomorphism (feel free to use the classification of groups of pq elements we discussed in class).

6. Consider the action of S_5 on the set X of pairs $\{i, j\}$ of distinct elements $1 \leqslant i \neq j \leqslant 5$. Element $\sigma \in S_5$ sends $\{i, j\}$ to $\{\sigma(i), \sigma(j)\}$.

   (a) Is the action $S_5 \curvearrowright X$ transitive?

   (b) Compute the order of the stabilizer of the element $\{1, 2\} \in X$.

7. (10 points) (a) (5 points) Compute the number of Sylow 2-subgroups of S_4.

   (b) (5 points) Explicitly describe all of them (namely, specify elements by which they are generated).

8. (10 points) Let $(G, \cdot)$ be a commutative monoid. Show that an element $g \in G$ is invertible if and only if the map:

$$g\cdot \colon G \to G,\ x \mapsto gx$$

is surjective.

9. (10 points) Let G be a finite group. Consider the set of integer-valued functions on G:

$$R = \{f \colon G \to \mathbb{Z}\}.$$

Define the operations $+, \star$ on R as follows:

$$(f+g)(x) := f(x) + g(x),$$

$$(f \star g)(x) = \sum_{y \in G} f(xy^{-1})g(y).$$

Show that $(R, +, \star)$ is a ring.

10. Let R be a commutative ring containing $F = \mathbb{Z}/p\mathbb{Z}$ as a subring (here $p > 1$ is a prime number). Show that for any $x, y \in R$, we have:

$$(x + y)^p = x^p + y^p.$$