# MATH 122 PSET 8 DUE 11/25 BY THE END OF THE DAY

The total number of points for all problems is 100.

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

## 1. Problem 1

Let $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ be the space of two by two matrices defined as follows:

$$\mathrm{Mat}_{2\times 2}(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{C} \right\}.$$

We define operations $+, \cdot$ on $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a+a' & b+b' \\ c+c' & d+d' \end{pmatrix}.$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa'+bc' & ab'+bd' \\ ca'+dc' & cb'+dd' \end{pmatrix}.$$

a) (5 points) Show that $(\mathrm{Mat}_{2\times 2}(\mathbb{C}), +, \cdot)$ is a ring.

b) (5 points) To any matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ one can associate its *determinant* $\det A = ad - bc$.

Show that for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we have

$$A \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \det A & 0 \\ 0 & \det A \end{pmatrix}. \tag{1.1}$$

c) (10 points) Show that $A$ is invertible if and only if $\det A \neq 0$. Using (1.1) write an explicit formula for $A^{-1}$. The group $\mathrm{Mat}_{2\times 2}(\mathbb{C})^{\times}$ of invertible martices is denoted by $\mathrm{GL}_2(\mathbb{C})$ and is called *general linear group*.

Hint: to prove that $A$ is not invertible if $\det A = 0$ you can assume for the sake of contradiction that $A$ has an inverse. Then multiply (1.1) by $A^{-1}$, you will get a contradiction.

## 2. Problem 2

Let $i = \sqrt{-1} \in \mathbb{C}$ be an imaginary unit (a complex number such that $i^2 = -1$). Recall that every complex number can be uniquely written in the form $a + bi$ for some $a, b \in \mathbb{R}$.

Set

$$\mathbb{H} = \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R} \right\} \subset \mathrm{Mat}_{2\times 2}(\mathbb{C}).$$

a) (5 points) Show that $\mathbb{H} \subset \mathrm{Mat}_{2\times 2}(\mathbb{C})$ is *closed* under addition and multiplication operations. In particular, $\mathbb{H}$ is a ring. It is called *ring of quaternions*.

b) (10 points) Show that every nonzero element of $\mathbb{H}$ is invertible (i.e., $\mathbb{H}$ is a division ring).

Hint: use Problem 1c).

c) (5 points) Show that $\mathbb{H}$ is *not* a field (i.e., find two elements in $\mathbb{H}$ that do not commute with each other.).

d) (5 points) Set:

$$\mathbf{1} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{i} := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ \mathbf{j} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \mathbf{k} := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

Show that these elements satisfy the following relations:

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{k},\ \mathbf{j}\cdot\mathbf{i} = -\mathbf{k},\ \mathbf{j}\cdot\mathbf{k} = \mathbf{i},\ \mathbf{k}\cdot\mathbf{j} = -\mathbf{i},\ \mathbf{k}\cdot\mathbf{i} = \mathbf{j},\ \mathbf{i}\cdot\mathbf{k} = -\mathbf{j},\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}. \quad (2.1)$$

*Remark* 2.1. Actually, every element of $\mathbb{H}$ can be uniquely presented by a linear combination of $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$ and relations (2.1) uniquely determine the product structure on $\mathbb{H}$.

e) (5 points) Write a general formula for the inverse to a nonzero element $a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, compute inverses of elements:

$$1 + \mathbf{i},\ \mathbf{i} + \mathbf{j},\ \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

## 3. Problem 3

(20 points) Let $R$ be a ring. Feel free to assume that $R$ is commutative or even $R = \mathbb{Z}$.

Show that $(R[x], +, \cdot)$ is a ring.

## 4. Problem 4

Let $k \in \mathbb{Z}_{>0}$ be a positive number. We define:

$$\varphi(k) := \#\{m = 1, \dots, k \mid \gcd(m, k) = 1\}.$$

a) (5 points) Compute $\varphi(1)$, $\varphi(2)$, $\varphi(3)$, $\varphi(4)$, $\varphi(5)$, $\varphi(6)$.

b) (5 points) Show that $\varphi(k)$ is equal to the number of elements $x \in \mathbb{Z}/k\mathbb{Z}$ such that $\langle x \rangle = \mathbb{Z}/k\mathbb{Z}$ (i.e., $x$ generates $(\mathbb{Z}/k\mathbb{Z}, +)$).

Hint: $x$ generates $\mathbb{Z}/k\mathbb{Z}$ iff $1 \in \langle x \rangle$. Use that $\varphi(k) = |(\mathbb{Z}/k\mathbb{Z})^\times|$.

c) (5 points) Show that $x \in \mathbb{Z}/k\mathbb{Z}$ generates $(\mathbb{Z}/k\mathbb{Z}, +)$ iff $x$ has order equal to $k$.

d) (5 points) Fix a positive number $n$ and let $k$ be a divisor of $n$. Show for an element $x \in \mathbb{Z}/n\mathbb{Z}$ the following two properties are equivalent:

1) $\underbrace{x + \ldots + x}_{k} = 0$

2) $x$ lives in the subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by $n/k$.

e) (5 points) Prove that the number of elements of order $k$ in $(\mathbb{Z}/n\mathbb{Z}, +)$ is equal to $\varphi(k)$.

Hint: use d) and then apply b)+c) to the subgroup of $\mathbb{Z}/n\mathbb{Z}$ generated by $n/k$.

f) (5 points) Show that the following equality holds:

$$\sum_{k|n} \varphi(k) = n.$$

Hint: every element in $\mathbb{Z}/n\mathbb{Z}$ has order equal to some $k|n$.