MATH 122 PSET 7 DUE 11/16 BY THE END OF THE DAY

The total number of points for all problems is 100.

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

1. Problem 1

a) (5 points) Find all Sylow 5-subgroups of the group D_{20}.

b) (5 points) How many elements of order 5 group D_{20} contains?

2. Problem 2

a) (5 points) Prove that there exists a unique homomorphism:

$$\varphi\colon \mathbb{Z}/4\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$$

such that $1 \mapsto (m \mapsto 2m)$.

Hint: use generators/relations presentation of $\mathbb{Z}/4\mathbb{Z}$.

b) (5 points) Using φ, we can define the semidirect product $(\mathbb{Z}/4\mathbb{Z}) \ltimes (\mathbb{Z}/5\mathbb{Z})$. Count the number of Sylow 5-subgroups in $(\mathbb{Z}/4\mathbb{Z}) \ltimes (\mathbb{Z}/5\mathbb{Z})$.

Remark: in this problem you are not required to explicitly construct Sylow 5-subgroups in D_{20} but it would be great if you do this.

c) (5 points) Count the number of elements of order 5 in $(\mathbb{Z}/4\mathbb{Z}) \ltimes (\mathbb{Z}/5\mathbb{Z})$.

Remark: in this problem you are not required to explicitly construct elements of order 5 in $(\mathbb{Z}/4\mathbb{Z}) \ltimes (\mathbb{Z}/5\mathbb{Z})$ but it would be great if you do this.

d) (5 points) Now let G be an arbitrary group of order 20. How many Sylow 5-subgroups can be contained in G?

Hint: use third Sylow theorem.

e) (10 points) How many elements of order 5 might be contained in a group of order 20?

3. Problem 3

The goal of this problem is to classify groups of order 10. Let G such a group.

a) (5 points) Using third Sylow theorem show that there is a *unique* Sylow 5-subgroup $H \subset G$.
b) (5 points) Show that H is a cyclic normal subgroup of G.
c) (5 points) Using Cauchy theorem (or Sylow theorem) show that G contains a cyclic subgroup of order 2. Let's denote it by K.
d) (5 points) Show that $H \cap K = \{1\}$.

e) (5 points) Show that elements of H, K generate the whole G.

Hint: use Lagrange's theorem and the fact that $[G : H] = 2$.

f) (10 points) Recall that H, K are cyclic. Let x be a generator of H and let y be a generator of K. Prove that G is generated by x, y satisfying relations:

$$x^5 = y^2 = 1, \ yxy^{-1} = x^r$$

for $r \in \{1, -1\}$.

Hint: recall that $H \subset G$ is normal, so we have a homomorphism

$$\mathbb{Z}/2\mathbb{Z} \simeq K \to \mathrm{Aut}(H) \simeq \mathrm{Aut}(\mathbb{Z}/5\mathbb{Z})$$

given by the conjugation $k \mapsto (h \mapsto khk^{-1})$.

g) (10 points) Prove that if $r = 1$, then G is isomorphic to

$$H \times K \simeq (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}).$$

h) (10 points) Prove that if $r = -1$, then G is isomorphic to

$$(\mathbb{Z}/2\mathbb{Z}) \ltimes (\mathbb{Z}/5\mathbb{Z}) \simeq D_{10},$$

where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/5\mathbb{Z}$ is induced by the map $1 \mapsto (m \mapsto -m)$.

i) (5 points) Prove that $\mathbb{Z}/10\mathbb{Z}$ is isomorphic to $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Hint: by the above results, it is enough to show that $(\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ is *not* isomorphic to D_{10}. Alternatively, you can construct an explicit isomorphism $\mathbb{Z}/10\mathbb{Z} \simeq (\mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

So, we see that Sylow theorems allow us to *classify* groups of order 10. Actually, using Sylow theorems one can classify groups of order pq, where $p < q$ are *arbitrary* prime numbers. All of them turn out to be isomorphic to semidirect products of $(\mathbb{Z}/p\mathbb{Z}) \ltimes (\mathbb{Z}/q\mathbb{Z})$ for appropriate homomorphisms $\mathbb{Z}/p\mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}/q\mathbb{Z})$.

You can try to prove this as follows (compare with Problem 3 above):

1) Using third Sylow theorem show that there is a unique Sylow q-subgroup $H \subset G$, show that H is normal and is isomorphic to $\mathbb{Z}/q\mathbb{Z}$.

2) Using first Sylow theorem prove that there exists a subgroup $K \subset G$ that is isomorphic to $\mathbb{Z}/p\mathbb{Z}$.

3) Prove that the map $\varphi\colon K \to \mathrm{Aut}(H)$ given by $k \mapsto (h \mapsto khk^{-1})$ is a homomorphism so we can form the semidirect product $K \ltimes H$.

4) Show that the map $K \ltimes H \to G$ given by $(k, h) \mapsto kh$ is a homomorphism. Prove that $H \cap K = \{1\}$ and deduce that the homomorphism $K \ltimes H \to G$ is injective. Conclude that it must be an isomoprhism.