

MATH 122 PSET 6 DUE 11/6 BY THE END OF THE DAY

The total number of points for all problems is 101. The maximum score you can earn on this problem set is 100. Therefore, if you solve problems worth a total of x points, your score will be $\min(x, 100)$.

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

1. PROBLEM 1

Consider group D_8 acting naturally on the square. It acts on the set $\{1, 2, 3, 4\}$ of vertices and on the set $\{[12], [23], [34], [41]\}$ of edges. It also acts on the set of two elements consisting of the diagonal lines.

- (10 points) Is the action on vertices transitive? Describe the stabilizer of the vertex 1.
- (10 points) Is the action on edges transitive? Describe the stabilizer of the edge $[12]$.
- (10 points) Is the action on diagonals transitive? Describe the stabilizer of a diagonal.

Hint: orbit–stabilizer theorem is very helpful to compute orders of stabilizers above.

2. PROBLEM 2

Consider the action of S_4 on the set $\{\{1, 2\} \sqcup \{3, 4\}, \{1, 3\} \sqcup \{2, 4\}, \{1, 4\} \sqcup \{2, 3\}\}$.

- (5 points) Prove that this action is transitive.
- (5 points) Describe the stabilizer of $\{1, 2\} \sqcup \{3, 4\}$.

3. PROBLEM 3

Let G be a group of order p^2 , where p is some prime number. The goal of this problem is to prove that G must be commutative.

We will prove on Monday (11/3) that if a group H contains p^k elements then its center is nontrivial. This result tells us that the center $Z \subset G$ is nontrivial. Our goal is to check that $Z = G$.

- (5 points) Assume for the sake of contradiction that $Z \neq G$. Show that $Z \subset G$ is normal and compute the number of elements in G/Z .
- (5 points) Prove that G/Z is cyclic.

c) (5 points) Prove that G must be commutative if G/Z is cyclic.

4. PROBLEM 4

Let G, H be a pair of groups. Let's define their *product*

$$H \times G = \{(h, g) \mid h \in H, g \in G\}$$

with the multiplication given by the formula:

$$(h_1, g_1) \cdot (h_2, g_2) = (h_1 h_2, g_1 g_2).$$

a) (5 points) Show that $(H \times G, \cdot)$ is indeed a group.

b) (10 points) Prove that there exists an isomorphism $(\mathbb{Z}/12\mathbb{Z})^\times \simeq (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

Hint: we classified groups of order 4.

Now, let's fix a homomorphism $\varphi: H \rightarrow \text{Aut}(G)$.

c) (5 points) Prove that the homomorphism φ defines the action $H \curvearrowright G$ given by the following formula:

$$h \cdot g = (\varphi(h))(g), \quad h \in H, \quad g \in G.$$

d) (10 points) Consider the following set:

$$H \rtimes G := \{(h, g) \mid h \in H, g \in G\}$$

and define the group structure on it by the following formula:

$$(h_1, g_1) \cdot (h_2, g_2) := (h_1 h_2, (h_2^{-1} \cdot g_1) g_2).$$

Prove that $(H \rtimes G, \cdot)$ is indeed a group. This group is called the *semidirect product* of G and H .

e) (5 points) Consider $\varphi: \mathbb{Z}/2\mathbb{Z} \rightarrow \text{Aut}(\mathbb{Z}/n\mathbb{Z})$ given by:

$$\mathbb{Z}/2\mathbb{Z} \ni 1 \mapsto (m \mapsto -m) \in \text{Aut}(\mathbb{Z}/n\mathbb{Z}),$$

$$\mathbb{Z}/2\mathbb{Z} \ni 0 \mapsto (m \mapsto m) \in \text{Aut}(\mathbb{Z}/n\mathbb{Z}).$$

Prove that φ is a homomorphism.

f) (10 points) Construct an isomorphism between D_{2n} and $(\mathbb{Z}/2\mathbb{Z}) \rtimes (\mathbb{Z}/n\mathbb{Z})$.

g)* (1 point) Prove that $G = \{(1, g) \mid g \in G\} \subset H \rtimes G$ is normal, prove that the quotient $(H \rtimes G)/G$ is isomorphic to H . You can assume that G is finite in this problem.