

MATH 122 PSET 4 DUE 10/9 BY THE END OF THE DAY

The total number of points for all problems is 100. The maximum score you can earn on this problem set is 105. Therefore, if you solve problems worth a total of x points, your score will be $\min(x, 100)$.

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

1. PROBLEM 1

(a) (10 points) Let H be the cyclic subgroup of the group A_4 generated by the permutation (123). Describe the left cosets A_4/H explicitly.

(b) (10 points) Prove that every subgroup of index 2 is a normal subgroup, and show by example that a subgroup of index 3 need not be normal.

(c) (15 points) Prove that A_4 has **no** subgroup of order 6.

Hint: use (b) and Lagrange's theorem

2. PROBLEM 2

(10 points) Let $\varphi: G \rightarrow G'$ be a group homomorphism. Suppose $|G| = 18$, $|G'| = 15$, and that φ is not the trivial homomorphism. What is the order of the kernel of φ ?

3. PROBLEM 3

(15 points) A group of order 22 contains elements x, y where $x \neq 1$, and y is not a power of x . Prove that x, y generate the whole group.

4. PROBLEM 4

(15 points) Let p be a prime number and consider the group $(\mathbb{Z}/p\mathbb{Z})^\times$. Prove that for every element $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ we have $a^{p-1} = 1$. Deduce the following fact: if $n \in \mathbb{Z}$ is an integer element not divisible by p , then $n^{p-1} \equiv 1 \pmod{p}$ (this is called Fermat's little theorem).

5. PROBLEM 5

(15 points) Let S be a *subset* of a group G that contains an identity element 1, and such that the left cosets aS , with a in G , partition G . Prove that S is a *subgroup* of G .

Hint: first prove that if $b \in S$, then $b^{-1} \in S$, then prove that if $a, b \in S$, then $ab \in S$ (use that by our assumptions two cosets xS and yS are either equal or disjoint).

6. PROBLEM 6

(15 points) Let G be a group and $H \subset G$ is a subgroup. Instead of considering the set of *left* cosets $G/H = \{gH \mid g \in G\}$ one can consider the set of *right* cosets $H \backslash G = \{Hg \mid g \in G\}$. Construct a bijection $G/H \xrightarrow{\sim} H \backslash G$ (note that we are not assuming that G, H are finite in this problem).