MATH 122 PSET 3 DUE 10/2 BY THE END OF THE DAY

The maximum score you can earn on this problem set is 100.

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

1. Problem 1

Find all automorphisms of:

(a) (10 points) a cyclic group of order 10;
(b) (10 points) the symmetric group S_3.

2. Problem 2

Let G be a group and $H \subset G$ be a subgroup. We will say that H is *normal* if for any $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$.

(a) (10 points) In the second homework, we introduced the group G of maps $f_{a,b} \colon \mathbb{R} \to \mathbb{R}$ of the form $f_{a,b} \colon x \mapsto ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$, and two subgroups of G:

$$H = \{f_{a,b} \,|\, a = 1\}, \;\; K = \{f_{a,b} \,|\, b = 0\}.$$

Which of these two is normal in G?

(b) Consider the following subgroup of S_4:

$$K = \{1, (12)(34), (13)(24), (14)(23)\} \subset S_4.$$

Prove that $K \subset S_4$ is normal.

(c) (5 points) Let a be an element of some group G. Prove that if the set $\{1, a\}$ is a normal subgroup of G, then a commutes with every element of G (i.e., a lies in the *center* of G).

(d) (5 points) Let H be a subgroup of a group G. Pick any element $g \in G$. Prove that the subset $gHg^{-1} \subset G$ is a subgroup.

3. Problem 3

(a) (10 points) Let $\sigma_1, \sigma_2 \in S_n$ be two permutations. In terms of cycle notation, describe when these two permutations are conjugate (i.e., when there exists $g \in S_n$ such that $\sigma_2 = g\sigma_1 g^{-1}$).

(b) (5 points) We will say that two permutations $\sigma_1, \sigma_2 \in S_n$ are equivalent if they are conjugate. Compute the number of equivalence classes in S_5.

4. Problem 4

Let U denote the set of *upper triangular* 2×2 *matrices* $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b \in \mathbb{R}^\times$, $c \in \mathbb{R}$. In other words, U is the set of triples (a, b, c) with $a, b \in \mathbb{R}^\times,\ c \in \mathbb{R}$.

(a) (10 points) Prove that the following operation defines the group structure on U:

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix}.$$

(b) (10 points) Prove that the map $\varphi\colon U \to (\mathbb{R}^\times, \cdot)$ given by

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mapsto a^2$$

is a homomorphism and determine its kernel and image.

5. Problem 5

(a) (10 points) Let $f\colon (\mathbb{R}, +) \to (\mathbb{C}^\times, \cdot)$ be the map $f(x) = e^{ix}$. Prove that f is a homomorphism, and determine its kernel and image.

(b) (10 points) Describe all homomorphisms $(\mathbb{Z}, +) \to (\mathbb{Z}, +)$. Determine which are injective, which are surjective, and which are isomorphisms.

6. Problem 6

(5 points) Let $\varphi\colon G \twoheadrightarrow G'$ be a surjective homomorphism. Prove that if G is cyclic, then G' is cyclic, and if G is abelian, then G' is abelian.