MATH 122 PSET 2 DUE 9/26 BY THE END OF THE DAY

The total number of points for all problems is 120. The maximum score you can earn on this problem set is 100. Therefore, if you solve problems worth a total of x points, your score will be $\min(x, 100)$.

Problems with $*$ are more complicated.

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

1. Problem 1

Let G be a set of maps $f_{a,b}\colon \mathbb{R} \to \mathbb{R}$ of the form $f_{a,b}\colon x \mapsto ax + b$ for some $a, b \in \mathbb{R}$ with $a \neq 0$.

(a) (10 points) Show that G is a group, with group law given by composition.

(b) (10 points) Show that the subsets:

$$H = \{f_{a,b} \,|\, a = 1\},\ K = \{f_{a,b} \,|\, b = 0\}$$

are subgroups of G.

2. Problem 2

Let G be a group.

(a) (5 points) Show that the set of automorphisms of G is itself a group (with group law given by composition). This group is denoted $Aut(G)$.

(b) (5 points) For each element $a \in G$, define a map $c_a\colon G \to G$ by $c_a(x) = axa^{-1}$. Show that c_a is an automorphism of G.

(c) (10 points) Show that the map $\phi\colon G \to Aut(G)$ defined by sending $a \in G$ to $c_a \in Aut(G)$ is a homomorphism.

3. Problem 3

(a) (10 points) Let a, b be elements of a group G. Show that ab and ba have the same order.

(b) (10 points) How many elements of order 2 does the symmetric group S_4 contain?

4. Problem 4

For a permutation $\sigma \in S_n$ let's define its *length* $\ell(\sigma)$ as follows:

$$\ell(\sigma) = \#\{1 \leqslant i < j \leqslant n \,|\, \sigma(i) > \sigma(j)\}.$$

(a) (5 points) Show that $\ell(\sigma) = \ell(\sigma^{-1})$.

(b)* (10 points) Equip $\{\pm 1\}$ with a group structure given by multiplication. Show that the map

$$\mathrm{sign}\colon S_n \to \{\pm 1\}, \sigma \mapsto (-1)^{\ell(\sigma)}$$

is a group homomorphism.

Hint: prove that every element of S_n is a product of the elements of the form $(i, i+1)$. Show that $\ell(\sigma \cdot (i, i+1)) = \ell(\sigma) \pm 1$.

An alternative approach is the following. Imagine that you have points with coordinates $(1,0), (2,0), \ldots, (n,0)$ and also points with coordinates $(1,1), (2,1), \ldots, (n,1)$. Now, we connect $(i,1)$ with $(\sigma(i), 0)$ by an oriented curve and claim that $\ell(\sigma)$ has the same parity as the number of intersection points of these curves (we assume there are no triple intersections). Then you can check by drawing pictures that the parity of $\ell(\sigma_1\sigma_2)$ is equal to the parity of $\ell(\sigma_1) + \ell(\sigma_2)$.

(c) (5 points) We will say that a permutation $\sigma \in S_n$ is *even* if $\ell(\sigma)$ is even. Otherwise, we will say that σ is *odd*.

Show that the subset $A_n \subset S_n$ of even permutations is a subgroup (feel free to use part (b) even if you don't know how to solve it!).

(d) (5 points) Explicitly describe the group A_3.

5. Problem 5

We will say that a group G is finitely generated if there exists a finite collection $a_1, \ldots, a_n \in G$ of elements of G such that every element of G can be written as $a_1^{k_1} \ldots a_n^{k_n}$ for some $k_i \in \mathbb{Z}$. Elements $a_1, \ldots, a_n$ will be called *generators* of the group G.

(a) (5 points) Prove that the group $(\mathbb{Z}/12\mathbb{Z})^\times$ is generated by two elements. Write down the relations.

(b)* (10 points) Prove that the group S_4 is also generated by two elements.

Hint: take (12) and (1234).

(c) (10 points) Prove that the group $(\mathbb{Q}, +)$ of rational numbers is not finitely generated.

6. Problem 6

(10 points) Prove that every group G of order 4 is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ or to $(\mathbb{Z}/12\mathbb{Z})^\times$.

Hint: think about possible orders of the elements in G.