

## MATH 122 PSET 1 DUE 9/17 BY THE END OF THE DAY

The total number of points for all problems is 110. The maximum score you can earn on this problem set is 100. Therefore, if you solve problems worth a total of  $x$  points, your score will be  $\min(x, 100)$ .

I remind you that using AI to give you answers to or help you answer homework problems is just as much cheating and unethical and honor code violating as asking a person to do that. I trust you will hold yourself to the highest ethical standards!

### 1. PROBLEM 1

Let  $S$  be a group.

(a) (10 points) Let  $x_1, \dots, x_n \in S$  be any elements in  $S$ . Show that

$$(x_1 x_2 \dots x_n)^{-1} = x_n^{-1} \cdot x_{n-1}^{-1} \cdot \dots \cdot x_1^{-1}.$$

(b) (10 points) Using what we proved in the first lecture (see Proposition 2.1.4 in Artin's book) check that the product of any  $n$  elements in  $S$  does not depend on the order in which we take it. For example:

$$a_1((a_2 \cdot a_3) \cdot a_4) = (a_1 \cdot (a_2 \cdot a_3)) \cdot a_4.$$

### 2. PROBLEM 2

Recall that in Lecture 2 we defined two operations  $+$  and  $\cdot$  on the set

$$\mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \dots, n-1\}.$$

- (a) (5 points) Check that  $(\mathbb{Z}/n\mathbb{Z}, +)$  is a group.
- (b) (5 points) Make multiplication tables for  $(\mathbb{Z}/4\mathbb{Z}, +)$  and  $(\mathbb{Z}/5\mathbb{Z}, +)$ .
- (c) (5 points) Check that  $(\mathbb{Z}/n\mathbb{Z}, \cdot)$  is a monoid.
- (d) (5 points) Make multiplication tables for  $(\mathbb{Z}/4\mathbb{Z}, \cdot)$  and  $(\mathbb{Z}/5\mathbb{Z}, \cdot)$ .

### 3. PROBLEM 3

Let  $S$  be a monoid. Recall that:

$$S^\times = \{x \in S \mid x \text{ is invertible}\}.$$

- (a) (10 points) Prove that  $S^\times$  is a group.
- (b) (10 points) Prove that

$$(\mathbb{Z}/n\mathbb{Z})^\times = \{k \in \mathbb{Z}/n\mathbb{Z} \mid \gcd(k, n) = 1\}.$$

Hint: use Proposition 2.3.5 in Artin's book (it will be covered in Lecture 4), see also Corollary 2.3.6 in the book.

- (c) (5 points) Make the multiplication table for  $(\mathbb{Z}/12\mathbb{Z})^\times$ .

## 4. PROBLEM 4

(a) (10 points) Show that a group  $S$  cannot be the union of two proper subgroups.

Hint: try to multiply two elements of these proper subgroups!

(b) (10 points) Explicitly describe *all* groups  $S$  that contain no proper *nontrivial* subgroup (by trivial subgroup we mean the one consisting of the identity element).

Hints:

1) Considering cyclic subgroups of  $S$  (see Proposition 2.4.2 in Artin's book) reduce the question to the case  $S = \mathbb{Z}/n\mathbb{Z}$ .

2) Prove that  $S = \mathbb{Z}/n\mathbb{Z}$  contains no proper *nontrivial* subgroups iff  $n$  is prime (Corollary 2.3.6 in the book can help).

More hints to 2):

If  $n = kl$  for some  $1 < k, l < n$ , then prove that the subgroup of  $\mathbb{Z}/n\mathbb{Z}$  generated by  $k$  will be proper nontrivial.

Now, if  $n = p$  is prime prove that for any nonzero  $x \in \mathbb{Z}/n\mathbb{Z}$  the cyclic subgroup  $\langle x \rangle$  of  $\mathbb{Z}/n\mathbb{Z}$  generated by  $x$  contains 1 (and hence coincides with  $\mathbb{Z}/n\mathbb{Z}$ ). To prove that  $\langle x \rangle$  contains 1 use that since  $x$  is coprime to  $p$ , there exist  $r, s \in \mathbb{Z}$  such that  $rx + sp = 1$ .

## 5. PROBLEM 5

(a) (5 points) Make the multiplication table for the group  $S_3$ .

(b) (10 points) How many subgroups does  $S_3$  have?

Hint: consider two cases – subgroup contains a 3-cycle or it doesn't.

## 6. PROBLEM 6

Let  $S, S'$  be two groups. A map

$$\varphi: S \rightarrow S'$$

is called *homomorphism* if

$$\varphi(ab) = \varphi(a)\varphi(b)$$

for any  $a, b \in S$ .

(10 points) Let  $S$  be a group and consider the map  $\varphi: S \rightarrow S$  sending each element  $a$  to its square  $a^2 \in S$ . Show that  $\varphi$  is a homomorphism if and only if  $S$  is commutative.

Hint: use the *cancellation law* (See Proposition 2.2.3 in Artin's book) claiming that if  $ab = ac$  then  $b = c$  and vice versa. Do you know how to prove the cancellation law?