Lecture 9

Last time

$$H \lhd G \quad \text{normal if } \forall g \in G,\ gHg^{-1} = H$$

$$G/H = \{gH \mid g \in G\}$$

set of *equivalence classes*, for $a \sim b \iff a^{-1}b \in H$.

Question: When does G/H have a group structure?

It's tempting to say that G/H is a group with a product given by:

$$(aH)(bH) = (abH) \quad (*)$$

The problem is that this procedure is *not* well-defined in general.
Example: $G = S_3,\ H = \{1, (12)\}$

$$(123)H \cdot (132)H = H \quad \text{problem!}$$

$$(13)H \cdot (132)H = (23)H$$

Claim: If $H \lhd G$ is *normal*, then G/H has a group structure given by formula (*).
Proof: Need to check that the product (*) is well-defined.
In other words, need to check that if $a_1 \sim b_1,\ a_2 \sim b_2$, then $a_1a_2 \sim b_1b_2$.
Indeed $b_1 = a_1h_1,\ b_2 = a_2h_2$ for some $h_1, h_2 \in H$.
Then

$$b_1b_2 = a_1h_1a_2h_2 = a_1a_2(a_2^{-1}h_1a_2)h_2 \in H$$

(using that H is normal).
Exercise: If formula (*) defines the group structure on G/H, then $H \lhd G$ is normal.

Upshot: If $H \leq G$ is a subgroup, then G/H as a set becomes a group if $H \lhd G$.

Example: $G = S_3,\ H = A_3 = \{1, (123), (132)\}$
Then G/H consists of two elements: $\{H, (12)H\}$, which is isomorphic to $\{\pm 1\}$.
If $H \lhd G$, then

$$G \to G/H, \quad g \mapsto gH$$

is a surjective homomorphism with kernel H.

Theorem: If $\varphi : G \to G'$ is a surjective homomorphism and $H = \ker \varphi$, then 1) $G/H \cong G'$ 2) The following diagram is commutative:

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G' \\ \downarrow & & \uparrow f \\ G/H & \xrightarrow{\cong} & G' \end{array}$$

In other words, φ "identifies" G with G/H.

Proof: Let's construct a map $f : G/H \to G'$.

Elements of G/H are cosets gH. Define

$$f(gH) := \varphi(g)$$

Note that $\varphi(H) = \{1\}$, so f is well-defined (does not depend on the choice of g).

By definition, the diagram is commutative. Remains to check that f is an isomorphism.

1) f is a homomorphism: take aH, bH.

We want $f((aH)(bH)) = f(aH)f(bH)$.

$$f(abH) = \varphi(ab) = \varphi(a)\varphi(b)$$

Indeed equal.

2) f is surjective: follows from commutativity of the diagram.

3) f is bijective: enough to check that $\ker f = \{1_{G/H}\}$.

Note now that $gH \mapsto 1$ iff $\varphi(g) = 1$ iff $g \in H$, i.e. $gH = H$.

Back to cosets:

Assume now that $H \leq G$ is an arbitrary subgroup. We assume G is finite.

Define

$$[G : H] := \#(G/H)$$

= number of left cosets gH. This number is called the *index*.

Lemma. All left cosets of a group have the same number of elements.

Proof: Enough to construct a bijection $aH \cong H$. It is given by left multiplication by a, inverse is left multiplication by a^{-1}.

Corollary.

$$|G| = |H| \cdot [G : H]$$

Order of G (sometimes denoted $\#G$) equals size of one coset times the number of cosets.