

9.29.2025: Math 122 Lecture 8 Notes

Vasily Krylov

Last Time

By $\varphi : G \rightarrow G'$, we denote a homomorphism. Last time, we defined the image of φ to be $\text{im}\varphi = \{\varphi(g) | g \in G\} \subset G'$ and the kernel to be $\ker \varphi = \{g \in G | \varphi(g) = 1\} \subset G$. These are each subgroups within G' and G , respectively.

We also established that φ is injective if and only if $\ker \varphi = \{1_G\}$. Additionally, if G, G' are groups and $\varphi : G \rightarrow G'$ is an arbitrary homomorphism, then we have the formula $|G| = |\ker \varphi| |\text{im}\varphi|$. This only requires that $\text{im}\varphi$ is finite. (Here, $|G|$ denotes the order of the group G ; one can alternatively write $\#G$).

Homomorphism Between Symmetric Groups

There is a homomorphism $\varphi : S_4 \rightarrow S_3$ defined as follows. Let $\Pi_1 = \{1, 2\} \cup \{3, 4\}$, $\Pi_2 = \{1, 3\} \cup \{2, 4\}$, $\Pi_3 = \{1, 4\} \cup \{2, 3\}$. Any $\sigma \in S_4$ permutes Π_1, Π_2, Π_3 ; e.g. $\sigma(\Pi_1) = \{\sigma(1), \sigma(2)\} \cup \{\sigma(3), \sigma(4)\}$. If $\sigma = (23)$, then $\sigma(\Pi_1) = \{1, 3\} \cup \{2, 4\} = \Pi_2$, $\sigma(\Pi_2) = \Pi_1$, $\sigma(\Pi_3) = \Pi_3$. This gives a homomorphism $\varphi : S_4 \rightarrow S_{\{\Pi_1, \Pi_2, \Pi_3\}} = S_3$. For example, we showed that $(23) \mapsto (12)$.

What is $\text{im}\varphi$? We claim that φ is surjective, i.e. $\text{im}\varphi = S_3$. Since $(12) \in \text{im}\varphi$, it suffices to show that $(123) \in \text{im}\varphi$, as $(12), (123)$ generate S_3 . Since $\varphi(132) = (123)$ (one can check this), it follows that φ is surjective. We conclude from this that $|\ker \varphi| = \frac{|S_4|}{|S_3|} = 4$. We observe that $\ker \varphi = \{1, (12)(34), (13)(24), (14)(23)\}$. This subgroup is isomorphic to $(\mathbb{Z}/12\mathbb{Z})^\times$. We have just given an embedding $(\mathbb{Z}/12\mathbb{Z})^\times \hookrightarrow S_4$.

Kernels and Normal Subgroups

In some sense, one can recover a surjective homomorphism $\varphi : G \rightarrow G'$ as well as G' from knowing the kernel $\ker \varphi \subset G$. First, we note that not every subgroup $H \subset G$ can be obtained as $\ker \varphi$ for some φ – only normal subgroups!

Def. $H \subset G$ is normal if $\forall g \in G, h \in H$ we have $ghg^{-1} \in H$ (i.e. $\forall g \in G, gHg^{-1} = H$).

Claim. $\ker \varphi$ is a normal subgroup.

Proof. Let $h \in \ker \varphi, g \in G$. We have that $\varphi(ghg^{-1}) = \varphi(g)\varphi(h)\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = \varphi(1) = 1$. This shows that $ghg^{-1} \in \ker \varphi$ as desired. \square

If G is any group, $\{1\}, G \subset G$ are always normal subgroups of G . These corresponds to the kernels of φ when φ is injective or when φ sends every element to 1, respectively.

Examples.

1. If G is commutative, then every subgroup of G is normal.
2. If $G = S_3$, the two-element subgroups are not normal, but the subgroup $A_3 = \{1, (123), (132)\}$ is normal, as it is the kernel of the sign homomorphism.

Idea. (Will become a theorem later). If $\varphi : G \rightarrow G'$, $H = \ker \varphi$, then

1. $G' \cong G/H := \{gH | g \in G\}$ (where this is an isomorphism of groups)
2. φ identifies with $G \rightarrow G/H, g \mapsto gH$. (Note that \rightarrow denotes a surjective map and \hookrightarrow denotes an injective one.)

Def. We now define the notion of a coset. If $H \subset G$ is an arbitrary subgroup, then for $g \in G$, we define the left coset of H for g to be $gH = \{gh | h \in H\}$. Note that H is a left coset for $g = 1$.

We can define an equivalence relation \sim on G . We let $a \sim b$ if $a^{-1}b \in H$, i.e. $b \in aH$. We see that $a \sim b$ if and only if $aH = bH$. Note that \sim has the following properties:

1. Transitivity: $a \sim b, b \sim c \implies a \sim c$
2. Reflexivity: $a \sim b \implies b \sim a$
3. $a \sim a$

Proof. We have that $a \sim b \iff a^{-1}b \in H, b \sim c \iff b^{-1}c \in H$, so $(a^{-1}b)(b^{-1}c) \in H \implies a^{-1}c \in H \implies a \sim c$.

Next, we see that $a \sim b \iff a^{-1}b \in H$. Since H is a subgroup, this implies $b^{-1}a \in H$, as it contains inverses.

Finally, $a^{-1}a = 1 \in H$ since H is a subgroup, so $a \sim a$. □

Exercise. An equivalence relation on X is the same as a partition of X into a disjoint union of equivalence classes. We have that (X, \sim) consists of $X = \sqcup(\text{equivalence classes for } \sim)$, where the equivalence class for $a \in X$ consists of all $b \in X$ with $a \sim b$. Starting with an equivalence relation, we can partition X into pieces, and starting with a set of equivalence classes, we can define an equivalence relation.

Going back to $X = G$, \sim given by $a \sim b \iff b \in aH$, we see that equivalence classes for \sim are cosets of the form aH . We also have a disjoint union $G = \sqcup aH$ of G into equivalence classes.

Example. Let $G = S_3, H = \{1, (12)\}$. We have three left cosets: $H = \{1, (12)\}, (123)H = \{(123), (13)\}, (132)H = \{(132), (23)\}$.

Comment. We have that $G/H = \{aH | a \in G\}$ is the set of equivalence classes for \sim . This is a group if and only if $H \subset G$ is normal.