

Math 122 – Lecture 23 Notes

1 Last Time

A vector space V is a module over a field F . On pset 9, we will prove that if V is finitely generated over F , say by $v_1, \dots, v_n \in V$, then $V \cong F \times \dots \times F = F^{\times n}$, where $n = \dim V$. Here, \cong denotes an isomorphism. For example, \mathbb{C}/\mathbb{R} has $\mathbb{C} \cong \mathbb{R}^2$.

2 More Example of Vector Spaces

Consider $\mathbb{H} = \left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$, the vector space of quaternions. We have $\mathbb{H} \cong \mathbb{R}^4$. The isomorphism is given by $\begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \mapsto (a, b, c, d)$. The target vector space has standard generators $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1)$, so we can obtain generators of \mathbb{H} by looking at their preimages. This yields the following generators of \mathbb{H} :

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

We call these $1, i, j, k$, respectively.

Next, consider \mathbb{H}/\mathbb{C} , where $\mathbb{C} = \{a + bi\} = \left\{ \begin{pmatrix} a+bi & 0 \\ 0 & a-bi \end{pmatrix} \right\}$ is a subfield of \mathbb{H} . We have $\mathbb{H} \cong \mathbb{C}^2$. The generators over \mathbb{C} are $1, j$. We see that $k = ij$ and $i \in \mathbb{C}$, and $i = i1$, so we get both i, k from $1, j$. This proves that $1, j$ are generators for \mathbb{H} over \mathbb{C} . We have that $\mathbb{C} \subset \mathbb{H}$ is a subfield of \mathbb{H} generated by $1, i$.

In general, if R is a ring that contains a field F as a subring, then $F \curvearrowright R$ and R becomes a vector space over F . We have that $R \curvearrowright R$ by left multiplication, and we simply restrict this action to F .

As another example, consider $R = F[x]/F$. The inclusion map $F \mapsto F[x]$ realizes F as a subfield of $F[x]$. This means $F \curvearrowright F[x]$ and $F[x]$ becomes a vector space. Is $F[x]$ finitely generated over F ? Equivalently, we ask if it is possible to find a collection p_1, \dots, p_m of polynomials such that any other polynomial is equal to $a_1 p_1 + \dots + a_m p_m$ for some choice of $a_1, \dots, a_m \in F$. Since the degree of $a_1 p_1 + \dots + a_m p_m$ is bounded by $\max(\deg p_i)$ but the degree of polynomials in $F[x]$ is unbounded, it is impossible to find a finite generating set

for $F[x]$ over F . This proves that $F[x]$ is not finitely generated over F .

Next, consider $F \curvearrowright F[x]$, and look at the subset $F[x]^{\leq n} = \{\text{polynomials of deg } \leq n\}$. This is not closed under multiplication, but it is a vector subspace of $F[x]$, as it is closed under addition and scalar multiplication. It is also of finite dimension, with generators $1, x, \dots, x^n$ and dimension $n + 1$.

Consider the map $F[x] \hookrightarrow F \times \dots \times F$, where $x^n \mapsto (0, \dots, 0, 1, 0, \dots)$, with a 1 in the $n + 1$ st entry. This is only an inclusion because every polynomial has only finitely many nonzero entries. For example, the element $(1, 1, 1, \dots)$ is not in the image of the map. Thus, $F[x]$ is isomorphic to the subset of $F \times \dots \times F$ where only finitely many entries are nonzero.

Remark. Can make $F[x]$ larger, namely $F[[x]] = \{a_0 + a_1x + a_2x^2 + \dots\}$ is the vector space of all Taylor series. We have $F[[x]] \cong F \times F \times \dots$.

3 Modules Over Rings

We want to understand modules over $F[x]$. If $F \subset F[x] \curvearrowright V$, then there is an action $F \curvearrowright V$ obtained from restricting the action. Thus V is a vector space over F with some additional structure. A module over $F[x]$ is:

1. A vector space V over F , i.e. $F \curvearrowright V$
2. A map $(x \cdot -) : V \rightarrow V$, with $v \mapsto x \cdot v$. Call this map f .

The map f has certain properties. We have $x(v_1 + v_2) = xv_1 + xv_2 \iff f(v_1 + v_2) = f(v_1) + f(v_2)$. We also have $f(av) = x(av) = axv = af(v)$ using commutativity, where $a \in F$. Thus $f(v_1 + v_2) = f(v_1) + f(v_2)$, $f(av) = af(v)$, so f is a linear transformation. An $F[x]$ -module is equivalent to a pair $(V, f : V \rightarrow V)$ where f is a linear transformation.

Assume we start with (V, f) . We define $F[x] \curvearrowright V$ as $a_0 \cdot v = a_0v, x \cdot v = f(v), x^2 \cdot v = f(f(v)), \dots$. Thus $(a_0 + a_1x + a_2x^2 + \dots)V = a_0v + a_1f(v) + a_2f^2(v) + \dots + a_n f^n(v)$.

As an example, let $V = \mathbb{R}^2, f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We have $f(a, b) = (b, 0)$. For example, $2 + 3x + 4x^2$ acts by $\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$, mapping $(a, b) \mapsto (2a + 3b, 2b)$.

Imagine now if $f = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ so $f : (a, b) \mapsto (2a, 3b)$. Now $2 + 3x + 4x^2$ acts by $(a, b) \mapsto (24a, 47b)$.

Let (V, f) be a module over $F[x]$. Then $W \subset V$ is an $F[x]$ -submodule if and only if $W \subset V$ is a vector space and $f(W) \subset W$.