Math 122 – Lecture 18 Notes

Last Time

Last time, we proved the first Sylow theorem, which states that if G is a finite group with order $p^k n$, then $\exists H \subset G$ with order p^k. We call H a Sylow p-subgroup.

Second Sylow Theorem

Today, we will prove the second Sylow theorem. This is as follows:

Theorem. (Second Sylow Theorem). Fix $H \subset G$ a Sylow p-subgroup. Then

(a) If $H' \subset G$ is another Sylow p-subgroup, then $\exists g \in G$ such that $H' = gHg^{-1}$.

(b) If $K \subset G$ is any subgroup, $\exists g \in G$ such that $K \subset gHg^{-1}$ (with $|K| = p^m$ here).

Example. Consider the example where $G = S_4, H = \langle (23)(14), (23) \rangle$, which is isomorphic to D_8, so is a Sylow 2-subgroup of G. We have another Sylow 2-subgroup given by $H' = \langle (12), (13)(24) \rangle$. Let $(142) = g \in S_4$. We claim that $H' = gHg^{-1}$.

Proof. We now prove the second Sylow theorem. We claim it suffices to prove the second part of the theorem. If we know part b), then if H' is another Sylow p-subgroup, we know $H' \subset gHg^{-1}$ for some $g \in G$. Since the order of each group is the same, i.e. $|H'| = p^k = |H| = |gHg^{-1}|$, it follows that $H' = gHg^{-1}$.

To prove b), consider $G \curvearrowright (G/H)$ which acts via $g(aH) = gaH$. Let $X = G/H$. If $K \subset G$ is any p-subgroup, then $K \curvearrowright X$ as well. Note that $|X| = \frac{|G|}{|H|} = n \implies p \not| |X|$.

We claim that $\exists x \in X$ such that $K_x = K$, i.e. K has a "fixed point". This follows from the general lemma:

Lemma. If $K \curvearrowright X$ with K a p-group and $p \not| |X|$, then $\exists x \in X$ such that $K_x = K$.

Proof. We have that $|X| = \sum [K : K_{x_i}]$, where x_i are representatives of K-orbits in X. There must exist $x_i = x$ such that $p \not| [K : K_{x_i}]$, since $p \not| |X|$. We claim that $K_x = K$. We have that $[K : K_x] = \frac{|K|}{|K_x|} = p^{m-l}$, as if $|K_x| < p^m$, then $|K_x| = p^l, l < m$. Since p^{m-l} cannot be divisible by p, we must have $m = l$, so $K_x = K$ as claimed. □

We have that $K = K_x \subset G_x$. Recall that $X = G/H \implies x = gH$ for some $g \in G$. We observe that $G_H = H, G_{gH} = gHg^{-1}$. If we can show this, it follows that $K \subset gHg^{-1}$, and we are done.

Why does gHg^{-1} stabilize gH? We have $ghg^{-1}gH = ghH = gH$. $\square$

Upshot: If G is finite with order $p^k n$, then

1. $\exists H \subset G$ with $|H| = p^k$
2. All such H are isomorphic and are of the form gHg^{-1} for some $g \in G$
3. If $K \subset G$ is any p-subgroup, then $K \subset gHg^{-1}$ for some $g \in G$.

Final question: How many of these H's do we have?

Theorem. (Third Sylow Theorem). Let $|G| = p^k n$, and let s be the number of Sylow p-subgroups. Then $s|n$, and $s \equiv 1 \mod p$.

Example. Let $G = S_3$, $p = 3, n = 2 \implies H = \langle (123) \rangle$. We see that $s|2, s \equiv 1 \mod 3 \implies s = 1$. If instead we take $H = \langle (12) \rangle$, then we have $H' = \langle (13) \rangle, H'' = \langle (23) \rangle \implies s \geq 3$. However, we know $s|3 \implies s = 3$.

Remark. If $g \in G$ and H is a Sylow p-subgroup, then $|gHg^{-1}| = |H| = p^k$, so $gHg^{-1} \subset G$ is a Sylow p-subgroup. We have that $s = 1 \iff H \subset G$ is normal.

Proof. We now prove the third Sylow theorem. Let X be the set of all Sylow p-subgroups. Let $G \curvearrowright X$ act by conjugation. We claim that G acts transitively. We have $X \cong G/G_H$, where G_H is the stabilizer of $H \in X$. We have that $G_H = \{g \in G | gHg^{-1} = H\}$, which is the normalizer of H. We have that $s = |X| = \frac{|G|}{|G_H|} | \frac{|G|}{|H|}$, as $H \subset G_H$ is a subgroup, so $|H|||G_H|$ by Lagrange. We have $\frac{|G|}{|H|} = n \implies s|n$.

Consider $H \curvearrowright X$, the restriction of the group action to H. We have that $H \in X$ is fixed by H. To show that $s \equiv 1 \mod p$, it suffices to show that no other $H' \in X$ is fixed by H (by the counting formula). If H fixes some H', then $\forall h \in H, hH'h^{-1} = H'$, so $H \subset G_{H'}$ is a Sylow p-subgroup, and we also note that $H' \subset G_{H'}$. Applying Sylow theorem to $G_{H'}$ yields that there exists some $g \in G_{H'}$ such that $H' = gH'g^{-1} = H$. $\square$