10.27.2025: Math 122 Lecture 15 Notes

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1 Last Time

Last time we established that $G \curvearrowright X$ partitions X into G orbits, so that $X = \sqcup \mathcal{O}$. Note that x and y are in the same orbit if $y = gx$ for some $g \in G$.

Example 1. $S_3 \curvearrowright S_3 = \{1\} \sqcup \{(12),(23),(13)\} \sqcup \{(123),(132)\}$, *where the action is conjugation and each of these is a conjugacy class.*

Moreover for $G \curvearrowright \mathcal{O}$ the action is transitive: $\forall\, x, y \in \mathcal{O}\, \exists\, g$ such that $y = gx$. So if $H \subset G$ (subgroup) then $G \curvearrowright G/H$, where the action is transitive. How does the action $G \curvearrowright G/H$ work? We have $g \in G$ and $aH \in G/H$, so $g \cdot (aH) = gaH$.

Example 2. $G = S_3$ *and* $H = \{1,(12)\}$. *Then* $G/H = \{\{1,(12)\},\{(23),(132)\},\{(13),(123)\}\}$, *where these are* C_1, C_2, C_3 *respectively. Then* $(12) \curvearrowright \{C_1, C_2, C_3\}$, *where conjugation permutes* C_1 *to itself and* C_2 *and* C_3 *between each other. To see this:*

$$(12) \cdot \{1,(12)\} = \{(12),1\} = C_1$$
$$(12) \cdot \{(23),(132)\} = \{(123),(13)\} = C_3$$
$$(12) \cdot \{(13) < (123)\} = \{(132),(23)\} = C_2.$$

Now note that $(123) \curvearrowright \{C_1, C_2, C_3\}$ *where* C_1 *permutes with* C_3, C_3 *goes to* C_2, *and* C_2 *goes to* C_1. *So moreover* $(123)C_1 = \{(123),(13)\} = C_3$, $(123)C_2 = C_1$, *and* $(123)C_3 = C_2$. *Check these!*

How does $(23) = (12)(123)$ *act? Then* $C_1 \xrightarrow{(123)} C_3 \xrightarrow{(12)} C_2$, $C_2 \xrightarrow{(123)} C_1 \xrightarrow{(12)} C_1$, $C_3 \xrightarrow{(123)} C_2 \xrightarrow{(12)} C_3$.

In general, why is $G \curvearrowright G/H$ or moreover $g \cdot (aH) = gaH$ indeed an action?

1. $1 \cdot (aH) = (1 \cdot a)H = aH$.
2. $g_1 \cdot (g_2 \cdot (aH)) = g_1 g_2 aH = g_1 g_2 \cdot (aH)$.

Note that $H \subset G$, and H need not be normal!

Claim 1. *Action* $G \curvearrowright G/H$ *is transitive (only one orbit).*

Proof. Every element of G/H is gH for some $g \in G$, now $gH = g \cdot H \to gH \sim H$. □

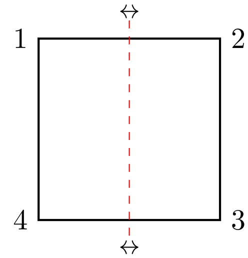
It turns out that any orbit $\mathcal{O}$ for $G \curvearrowright X$ can be identified with G/H for appropriate $H \subset G$. Namely, if $x \in \mathcal{O}$, we can define

$$G_x := \{g \in G \mid g \cdot x = x\} \subset G,$$

the stabilizer o G. We will see that $\mathcal{O} \simeq G/G_x$ as sets with the G-action.

Examples of stabilizers

$S_4 \curvearrowright \{1,2,3,4\}$: the stabilizer of $\{4\}$ is $S_3 \subset S_4$. We have the action of D_4 on



so that

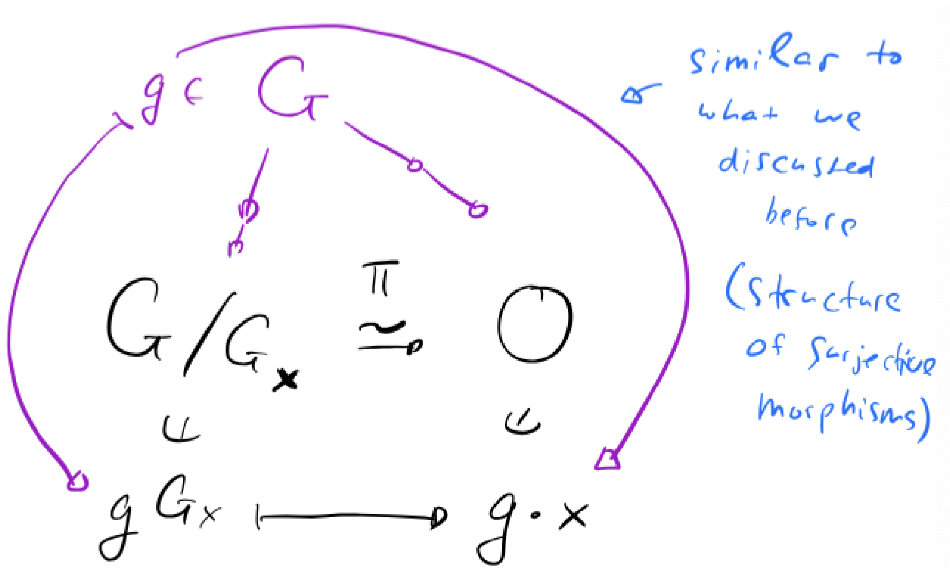
$$D_4 \curvearrowright \{[12],[14],[23],[34]\}$$

Then the stabilizer of $[12]$ is $\{1,(12)\}$, where note that $(12) = rs$. For $G \curvearrowright G/H$, the stabilizer of $H \in G/H$ is $H \subset G$ ($gH = H \to g \in H$).So we can "read off" H from $G \curvearrowright X = G/H$.

Claim 2. *$G_x \subset G$ is a subgroup.*

Proof. $1 \in G_x$ as $1 \cdot x = x$. If $g_1, g_2 \in G_x$ then $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = g_1 \cdot x = x$. □

If $\mathcal{O} \subset X$ (orbit) for $G \curvearrowright X$ and $x \in \mathcal{O}$, there exists an identification:



Proof. 1. $gG_x \mapsto g \cdot x$ is well defined. Take $a \in G_x$, then $(ga) \cdot x = g \cdot (a \cdot x) = gx$.

2. π is surjective: every element of $\mathcal{O}$ is of the form $g \cdot x$ for some $g \in G$ (using that $\mathcal{O}$ is an orbit).

3. π is injective: if $g_1 G_x \mapsto g_1 \cdot x$ and $g_2 G_x \mapsto g_2 \cdot x$ where $g_1 x = g_2 x$, then $(g_2^{-1} g_1) \cdot x = g_2^{-1} \cdot (g_1 \cdot x) = g_2^{-1} \cdot (g_2 \cdot x) = x$. Then $g_2^{-1} g_1 \in G_x \to g_2 G_x = g_2 \cdot (g_2^{-1} g_1) G_x = g_1 G_x$.

□