10.22.2025: Math 122 Lecture 14 Notes

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1 Last Time

Recall from last time that a group G acts on a set X if we are given a map $G \times X \to X$ that sends $(g, x) \to g \cdot x$ such that:
1.
$$1 \cdot x = x; \forall x \in X$$
2.
$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \ \forall g_1, g_2 \in G$$
For example, S_n acts on $\{1, 2, \cdots, n$ by $\sigma \cdot i = \sigma(i)$
Also D_{2n} acts on a $2n$-sided regular polygon X by $f \cdot x = f(x)$.

More generally: for any set X, S_x acts on X by;
$$\sigma \cdot x = \sigma(x)$$
Proof: $id \cdot x = id(x) = x$ and
$$\sigma_1 \cdot (\sigma_2 \cdot x) = \sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x)$$
More examples of actions: Consider S_4 acting on $\{1, 2, 3, 4\}$, but it also acts on the set;
$$\Big\{ \{1,2\} \sqcup \{3,4\}, \{1,3\} \sqcup \{2,4\}, \{1,4\} \sqcup \{2,3\} \Big\}$$
For example;
$$\begin{cases} (123) \cdot \Big(\{1,2\} \sqcup \{3,4\}\Big) = \Big(\{1,4\} \sqcup \{2,3\}\Big) \\ (123) \cdot \Big(\{1,3\} \sqcup \{2,4\}\Big) = \Big(\{1,2\} \sqcup \{3,4\}\Big) \\ (123) \cdot \Big(\{1,4\} \sqcup \{2,3\}\Big) = \Big(\{1,3\} \sqcup \{2,4\}\Big) \end{cases}$$
Remember that we used this to construct the homomorphism $S_4 \to S_3$.

Claim: G acting on X is the same as a homomorphism $G \to S_x$.

Proof: If G acting on $X \implies G \xrightarrow{\varphi} S_x$ by $g \to (x \to gx)$, φ is a homomorphism:
$$\varphi(g_1 g_2) =^? \varphi(g_1) \circ \varphi(g_2) \iff \varphi(g_1 g_2)(x) = \varphi(g_1)(\varphi(g_2)(x)) \forall x \in X$$
$$(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x)$$

In the opposite direction: If a homomorphism $\varphi : G \to S_x \implies$ G acts on X, $g \cdot x = \varphi(g)(x)$. Check that this formula indeed defines an action.

2 More Examples of Action

$G \hookrightarrow S_G$ (with $g \to (h \xrightarrow{m_g} gh) \implies G$ acts on itself by $g \cdot h = gh$
Ex: S_3 acts on S_3 with $(12) \in S_3$ by;

$$\begin{cases} 1 \to (12) \\ (12) \to 1 \\ (23) \to (123) \end{cases}$$

Also consider $G \hookrightarrow S_G$ by $g \to (h \to hg^{-1}) \implies G$ acts on itself by $g \cdot h = hg^{-1}$. Let's check that this is indeed an action.

$$g_1 \cdot (g_2 \cdot h) = g_1 \cdot (hg_2^{-1}) = hg_2^{-1}g_1^{-1} = h(g_1g_2)^{-1} = (g_1g_2) \cdot h$$

See that the inverse is crucial.

$G \to \mathrm{Aut}(G) \subset S_G$ where $g \to (h \to ghg^{-1}) \implies G$ acts on itself by $g \cdot h = ghg^{-1}$.
If $H \hookrightarrow G$ and G acts on $X \implies H$ acts on X, in particular H acts on G by;

$$h \cdot g = \begin{cases} hg \\ gh^{-1} \\ hgh^{-1} \end{cases}$$

3 Orbits and Stabilizers

Consider the following example, S_3 acts on itself by conjugation. Notice that $\{1\}$ is preserved by this action.

$$(12) \xrightarrow{(123)} (23) \xrightarrow{(12)} (13)$$

another conjugacy class $\{(12), (23), (13)\}$. Also $\{(123), (132)\}$ where the $(123) \xrightarrow{(12)} (132)$. We have that;

$$S_3 = \{1\} \sqcup \{(12), (23), (13)\} \sqcup \{(123), (132)\}$$

where each $\{1\}$, $\{(12), (23), (13)\}$, and $\{(123), (132)\}$ are orbits for S_3 acting on S_3 by conjugation.

In general, G acting on X partitions X into orbits. Namely, we say that x, y are equivalent $(x \sim y)$ if $\exists g \in G$ such that $y = gx$.

Claim: $\sim$ is an equivalence relation:

$$x \sim x; (x = 1 \cdot x)$$
$$x \sim y \implies y \sim x \text{ (if } y = g \cdot x \implies x = g^{-1} \cdot y)$$
$$x \sim y \sim z \implies y = g_1 x, z = g_2 y$$

so;

$$z = g_2 \cdot (g_1 x) = (g_2 g_1) \cdot x \implies x \sim z$$

We get that $X = \bigsqcup O$ where O is the equivalence classes for $\sim$, which is defined to be the "orbits" for the action.

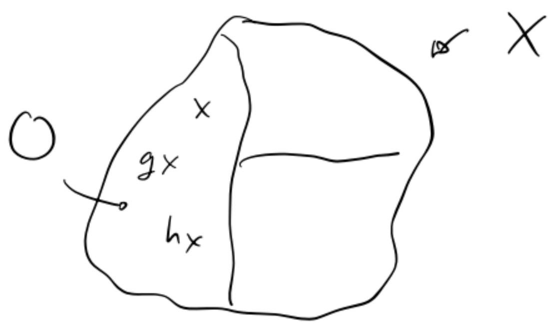

Figure 1: Orbits

Example: H acts on G by $h \cdot g = hg$ gives us that the H-orbits are right cosets Hg.
From earlier, S_3 acting on S_3 gave us;

$$\{1\} \sqcup \{(12), (23), (13)\} \sqcup \{(123), (132)\}$$

We see that many objects we introduced before are particular cases of this general structure.

Note: G acts on every orbit $O \subset X$; "identifies" all elments in this orbit. If G acts on X with one orbit, then this action is called **transitive**.

Ex: Consider G acting on G/H by $g \cdot (aH) = gaH$. Note that H need not be normal;
1) We have that $1 \cdot (aH) = (1 \cdot a)H = aH$. and
2) $g_1 \cdot (g_2 \cdot (aH)) = g_1 g_2 aH = (g_1 g_2) \cdot aH$.
The action G on G/H is transitive.

It turns out that any orbit O for G acting on X can be identified with G/H for appropriate $H \subset G$. Namely, if $x \in O$, we can define;

$$G_x := \{g \in G | g \cdot x = x\}$$

where G_x is the **stabilizer** of x. We will see that;

$$O \cong G/G_x$$

as sets with G-action.

4 Examples of stabilizers

Consider S_4 acting on $\{1, 2, 3, 4\}$. Stabilizer of $\{4\}$ is $S_3 \subset S_4$.

D_4 acting on the square with corners labeled $1, 2, 3, 4$ with reflection, so D_4 acting on $\{[12], [14], [23], [34]$, we see that the stabilizer of $[12]$ is $\{1, (12)\}$. For G acting on G/H, the stabilizer of $H \in G/H$ is $H \subset H$. So we can "read off" H from G acting on $X = G/H$.

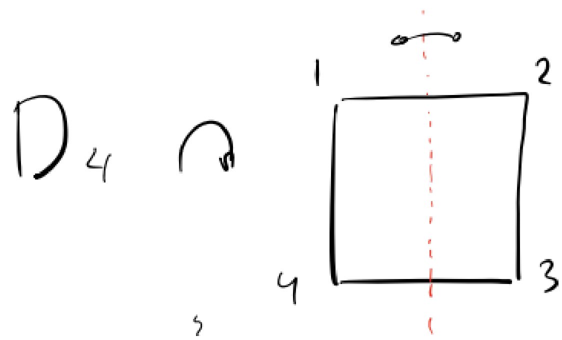

Figure 2: Action of D_4

Claim: $G_x \subset G$ is a subgroup.
Proof: $1 \in G_x$ as $1 \cdot x = x$. If $g_1 g_2 \in G_x$, then $(g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) = g_1 \cdot x = x$.

Proposition: If $O \subset X$ is an orbit for G acting on X and $x \in O$, then there exists an identification $G/G_x \cong^{\pi} O$ by $gG_x \to g \cdot x$, which is an identification of sets with G-action. We will prove this next time.