

10.20.2025: Math 122 Lecture 13 Notes

Vasily Krylov

1 Last Time

We defined $\langle x_1, \dots, x_n \rangle / \langle R \rangle_{norm}$. If $S = \{x_1, \dots, x_n\}$, we let $F(S) = \langle x_1, \dots, x_n \rangle$. We saw the main theorem that if $F : S \rightarrow S'$ is a homomorphism with kernel K , $N \subset S$ is a normal subgroup with $N \subset K$, then there is a unique homomorphism $h : S/N \rightarrow S'$ such that F factors through S/N , i.e. the following diagram commutes:

$$\begin{array}{ccc} S & \xrightarrow{F} & S' \\ & \searrow \varphi & \uparrow h \\ & & S/N \end{array}$$

We see that $gN \mapsto F(g)$ must be the case.

2 Proof

If we are given such a diagram, we claim that $\ker \varphi \subset \ker F$. This follows from the fact that h is a homomorphism and $F = h \circ \varphi$. We also see that $\ker \varphi = N$ by definition of the quotient map.

An arbitrary homomorphism F has the decomposition $S \twoheadrightarrow \text{im}F \hookrightarrow S'$. If $\ker F = K$, then $a : S/K \xrightarrow{\sim} \text{im}F$ is an isomorphism. We get the following diagram:

$$\begin{array}{ccc} S & \twoheadrightarrow \text{im}F & \hookrightarrow S' \\ & \searrow \pi & \uparrow a \\ & & S/K \end{array}$$

In other words, F identifies with the element-wise map $g \mapsto gK \mapsto F(g)$.

Next, consider when we have a normal subgroup $N \subset K$. We have the decomposition $S \twoheadrightarrow S/N \twoheadrightarrow S/K \hookrightarrow S'$ given by $g \mapsto gN \mapsto gK$, followed by inclusion into S' .

Remark. We have that $K/N \subset S/N$, and we have a map $S/N \rightarrow S/K$ given by taking the quotient of S/N by the normal subgroup K/N .

In the context of the decomposition $S \twoheadrightarrow S/N \twoheadrightarrow S/K \hookrightarrow S'$ we see that the map h must be given by $S/N \twoheadrightarrow S/K \hookrightarrow S'$.

Claim. We claim that $D_{2n} \cong \langle r, s \rangle / \langle r^n, s^2, (sr)^2 \rangle_{norm} = G$.

Proof. There exists a map $G \rightarrow D_{2n}$ given by $r \mapsto r \in D_{2n}, s \mapsto s \in D_{2n}$. Call it ψ . We first claim that ψ is surjective. We see that r, s generate D_{2n} and are in the image of ψ , which proves surjectivity.

Comment: If S is a set, H is a group, then you can always construct a map $F(S) \rightarrow H, x_i \mapsto y_i$ by picking $y_i \in H$. However, you cannot always construct a map $H \rightarrow F(S)$, e.g. if $y_i^2 = 1$, but no element of $F(S)$ has order 2.

We next claim that ψ is injective. It suffices to show that $|G| \leq 2n$. We have $|G| = |D_{2n}| |\ker \psi| = 2n |\ker \psi|$. Thus if $|G| \leq 2n$, it follows that $|\ker \psi| \leq 1$, i.e. $\ker \psi$ is trivial (or equivalently, ψ is injective).

Exercise: Show that every element of G is of the form r^k or sr^k with $0 \leq k \leq n-1$ (we do not need to show these are unique). Consider an arbitrary element $r^a s^b r^c \dots$. We have that $rsrs = 1 \implies rs = s^{-1}r^{-1} = sr^{-1}$. Using this relation, an arbitrary product $r^a s^b r^c \dots$ reduces to $s^k r^m$ for some k, m (as we are able to exchange the order of r and s via the relation $rs = sr^{-1}$, moving all s terms to the left and gathering all s and r terms together). Next, since s has order 2 and r has order n , it follows that $k = 0, 1$ and $0 \leq m \leq n-1$. \square

3 Group Action

Recall that D_{2n} was defined as the group of symmetries of the regular n -gon. We can say that D_{2n} acts on X (the n -gon), i.e. each $g \in D_{2n}$ can be applied to X as a transformation of the set X , i.e. $g : X \rightarrow X$. Any vertex $x \in X$ gets mapped to another vertex under the map $g(x)$.

The group S_n is the group of permutations of $\{1, 2, \dots, n\}$, so can be seen as acting on this set via $\sigma : i \mapsto \sigma(i)$.

Definition. We will say that a group G acts on a set X if we are given a map $G \times X \rightarrow X$ (here, $G \times X$ is the set of all pairs (g, x) with $g \in G, x \in X$). We can write the map as $(g, x) \mapsto g \cdot x = gx$. We also require that:

1. $1x = x$
2. $g_2(g_1x) = (g_1g_2)x$