# 10.15.2025: Math 122 Lecture 13 Notes

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## 1 Free Group

Start with an arbitrary set $S = \{x_1, \cdots x_n\}$ which may be infinite in general. $\mathcal{F}(S)$ is the free group generated by $x_i$, consists of all posssible products;

$$x_{i_1}^{\pm 1}, x_{i_2}^{\pm 1}, \cdots$$

modulo relations.

$$(*) \cdots a x_i x_i^{-1} b \cdots = \cdots ab \cdots$$

Now the formal definition of $\mathcal{F}(S)$ is given by $\widetilde{\mathcal{F}}(S) = \{x_{i_1}^{\pm 1}, x_{i_2}^{\pm 1}, \cdots\}$. We can define $\sim$ on $\widetilde{\mathcal{F}}(S)$ as follows:

Two elements of $\widetilde{\mathcal{F}}(S)$ are equivalent if one can be obtained from the other via operations $(*)$.

Claim: (see section 7.9 of Artin's book for details)
1. $\sim$ is the equivalence relation.
2. Every equivalence class has unique shortest representative (called reduced word), check this.
Example: $sts$ is reduced, whereas $sttt^{-1}s$ is not reduced.
Definition: $\mathcal{F}(S) := \widetilde{\mathcal{F}}(S)/\sim$ where this is the set of equivalence classes.

Claim: $\mathcal{F}(S)$ has a group structure given by: if $A, B \in \mathcal{F}(S)$, then to define $A \cdot B$ pick any $a \in A, b \in B$, then $A \cdot B$ is the equivalence class of $ab$. Check that this operation is well-defined. Is it clear what you need to check here?

## 2 Relations

Start with $\mathcal{F}(S)$, want to impose some relations $R \subset \mathcal{F}(S)$ (want all of these elements to be equal to 1). How to define $\mathcal{F}(S)/\langle R\rangle$?

Ex:

$$\langle r, s\rangle / \langle r^4, s^2, srsr\rangle$$

what does this quotient mean? Note that $r^4 = 1, s^2 = 1 \implies r^4 s^2 = 1 \implies$ the whole subgroup of $\mathcal{F}(S)$ generated by $R$ must be in $\langle R\rangle$.

Naively, can try $\mathcal{F}(S)$/subgroup generated by $R$. This is not a group in general, and the subgroup generated by $R$ may not be normal.

Correct thing to do is;

$$\langle R \rangle \subset \mathcal{F}(S)$$

where $\langle R \rangle$ is the minimal normal subgroup that contains $R$. And then $\mathcal{F}(S)/\langle R \rangle$ makes sense and has group structure.

How to think about $\langle R \rangle$ :
$\langle R \rangle$ are the elements of $G$ that can be obtained from $R$ using a finite sequence of the operations of multiplication, inversion, and conjugation.
Ex: $\langle r^4, s^2, srsr \rangle$ contains $rsrs = s^{-1}srsrs$.
The most important property of $\mathcal{F}(S)/\langle R \rangle$.
Theorem: Let $G$ be an arbitrary group, then;

$$\psi : \mathcal{F}(S)\langle R \rangle \to G$$

is the same as;
1. $y_i \in G \forall x_i \in S$ where $y_i$ is the collection of elements in $G$ such that;
2. $\forall x_{i_1} \cdots x_{i_k} \in R$, $y_{i_1} \cdots y_{i_k} = 1$.

Proof:
1. Given $y_i$, we can define;

$$\mathcal{F}(S) \xrightarrow{\tilde{\psi}} G$$

Maps $x_i \to y_i$, where it extends to a homomorphism (why?)
2. We also have;

$$\mathcal{F}(S) \xrightarrow{\varphi} \mathcal{F}(S)/\langle R \rangle$$

and want a map $\Psi : \mathcal{F}(S)/\langle R \rangle \to G$ that makes this diagram commutative. General theorem that implies 2;

Theorem: Let $f : S \to S'$ be a group homomorphism with kernel $K$. Let $N \subset S$ be normal and contained in $K$. Then there is a unique map $S/N \xrightarrow{h} S'$ (mapping $gN \to f(g)$) such that $S \xrightarrow{f} S'$, $S/N \xrightarrow{h} S'$ and $S \to S/N$ is commutative.

Proof: Consider the diagram;

$$S \to \mathrm{Im}(f) \hookrightarrow S'$$

with $f : S \to S'$ and additionally $\varphi : S \to S/K$ and $a : S/K \to \mathrm{Im}(f)$.

We proved that there exists an isomorphism;

$$a : S/K \xrightarrow{\sim} \mathrm{Im}(f)$$

That sends $gK \to f(g)$ such that the diagram is commutative. In other words, we have;

$$S \xrightarrow{\varphi} S/K \hookrightarrow S'$$

Sending $g \to gK \to f(g)$, with $f : S \to S'$. Using that $N \subset K$, we conclude that $\varphi$ factors through;

$$S \to S/N \to S/K \hookrightarrow S'$$

With $h : S/N \hookrightarrow S'$.

Back to our goal:

$$\mathcal{F}(S) \xrightarrow{\widetilde{\psi}} G$$
$$\varphi : \mathcal{F}(S) \to \mathcal{F}(S)/\langle R\rangle$$
$$\Psi : \mathcal{F}(S)/\langle R\rangle \to G$$

where we want $\Psi$ such that this is commutative, we apply the theorem to $f = \widetilde{\psi}, N = \langle R\rangle$. Remains to check that $\mathrm{Ker}(\widetilde{\psi})$ contains $\langle R\rangle$.
Indeed: 1. $\mathrm{Ker}(\widetilde{\psi})$ contains $R$ (remember that any element of $R$ becomes equal to 1 in $G$).
2. $\mathrm{Ker}(\widetilde{\psi})$ is normal.
These imply $\langle R\rangle \subset \mathrm{Ker}(\widetilde{\psi})$ as desired.

Claim:

$$D_{2n} = \langle r, s\rangle/\langle r^n, s^2, srsr\rangle$$

Proof: By the theorem above, the homomorphism;

$$\langle r, s\rangle/\langle r^n, s^2, srsr\rangle \xrightarrow{\psi} D_{2n}$$

is well-defined.
1. It is surjective since $r, s$ generate $D_{2n}$.
2. Remains to check that $\Big|\langle r, s\rangle/\langle r^n, s^2, srsr\rangle\Big| \leq 2n$ then $\psi$ must be an isomorphism (we proved that $|D_{2n}| = 2n$).

Exercise: Every element of $\langle r, s\rangle/\langle r^n, s^2, srsr\rangle$ is of the form $r^k$ or $sr^k \implies$ we have $\leq 2n$ elements in this group.