10.08.2025: Math 122 Lecture 11 Notes

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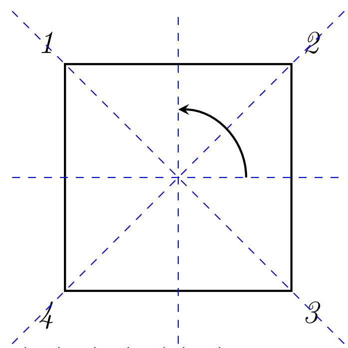
1 Last Time

We covered the dihedral groups D_{2n}, which are all symmetries of the n-gon. In particular, $D_{2n} \hookrightarrow S_n$, where we note that each symetry in D_{2n} merely permutes the n vertices. Moreover we can define an isomorphism from $D_6 \xrightarrow{\sim} S_3$.

Example 1. *Take $n = 4$, a square:*



It has 4 reflections: $s = (24), (12)(34), (13), (14)(23)$; and rotation by $\pi/2$ and its powers: $1, r = (1234), (13)(24), (1432)$. The following relation is clear: $r^4 = s^2 = 1$. Less obvious is the fact that

$$rs = sr^{-1} \Leftrightarrow rsr = s.$$

Theorem 1. *$D_{2n} = \langle r, s \rangle \,/\, r^n = s^2 = 1, rs = sr^{-1}$.*

Note that this presentation allows us to think about D_{2n} and homomorphisms from it in very explicit terms.

Claim 1. *We claim that*

1. *D_{2n} is generated by s, r.*
2. *$|D_{2n}| = 2n$.*

Proof. 1. Pick $\sigma \in D_{2n}$, where $1 \mapsto i$ for some i. Composing with r^{1-i}, we can assume that $\sigma : 1 \mapsto 1$. Then

$$n \mapsto \begin{cases} 2 \\ n \end{cases}$$

(where we have only these two options because σ preserves distances!) Composing with s (if needed), we can assume that $\sigma : 1 \mapsto 1, 2 \mapsto 2$, which determines σ uniquely (again use that σ preserves distances).

2. From the proof of 1) it is clear that

$$D_{2n} = \{1, r, r^2, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\},$$

which are all distinct. Then

$$n + 1 \leq |D_{2n}| \leq 2n,$$

so $|D_{2n}| = 2n$ (using the fact that $n \mid |D_{2n}|$).

□