Last time

$\varphi : G \to G'$ — homomorphism

- image of φ — $\{\varphi(g) \mid g \in G\}$
- kernel of φ — $\{g \in G \mid \varphi(g) = 1\}$

$\mathrm{im}\,\varphi \subset G'$; $\ker \varphi \subset G$

subgroups

- φ — injective $\iff \ker \varphi = \{1\}$

- if G, G' — finite $\Rightarrow \#G = \#\mathrm{im}\,\varphi \cdot \#\ker\varphi$

will use this equality today

Homomorphism $S_4 \to S_3$

Last time we also constructed a homomorphism

$$\varphi : S_4 \to S_3$$

Let me recall the construction of φ:

every element of S_4 acts on the set $\{\Pi_1, \Pi_2, \Pi_3\}$, where

$$\Pi_1 = \{1,2\} \cup \{3,4\}$$
$$\Pi_2 = \{1,3\} \cup \{2,4\}$$
$$\Pi_3 = \{1,4\} \cup \{2,3\}$$

do you understand why this is indeed a homomorphism?

So, we get a homomorphism:

$$S_4 \xrightarrow{\varphi} S_{\{\Pi_1, \Pi_2, \Pi_3\}} = S_3$$

For example,

$$\underset{\underset{S_4}{\rotatebox{90}{$\in$}}}{(23)} \longmapsto \underbrace{\begin{pmatrix} \Pi_1 \mapsto \Pi_2 \\ \Pi_2 \mapsto \Pi_1 \\ \Pi_3 \mapsto \Pi_3 \end{pmatrix}}_{=\,(12)\,\in\, S_3}$$

<u>Claim:</u> φ is surjective

<u>proof</u>: goal is check that $\underset{\text{group}}{\operatorname{im}\varphi} = S_3$

Enough to check that $\operatorname{im}\varphi$ contains <u>generators</u> of S_3. Recall that S_3 is generated by

$(12), (123)$

allready know that it lies in the image

Remains to check that $(123) \in \mathrm{im}(\varphi)$

Indeed:

$$(234) \mapsto \underbrace{\begin{pmatrix} \Pi_1 \mapsto \Pi_2 \\ \Pi_2 \mapsto \Pi_3 \\ \Pi_3 \mapsto \Pi_1 \end{pmatrix}}_{(123)}$$

So φ <u>is</u> surjective.

Let's now describe the kernel of φ.

$$\# \ker \varphi = \,?$$

Recall: $\# \ker \varphi = \dfrac{\# S_4}{\# \mathrm{im}\, \varphi} = \dfrac{\# S_4}{\# S_3} = 4$

So, to describe the kernel, remains to construct <u>4</u> elements of S_4 that act trivially on Π_1, Π_2, Π_3

It's easy to see that these elements

work: 1, $(12)(34)$, $(13)(24)$, $(14)(23)$

$\Downarrow$

$\ker\varphi = \{1, (12)(34), (13)(24), (14)(23)\}$

do you recognise this group?

(that's $(\mathbb{Z}/_{12}\mathbb{Z})^{\times}$!)

Normal subgroups

In some sense, one can <u>recover</u> surjective

homomorphism $\varphi : G \twoheadrightarrow G'$ as well

as G' from knowing the kernel $\ker\varphi$

as a <u>subgroup</u> of G

We will discuss how that works.

First step is the following observation.

Not **every** subgroup H of G can be obtained as a kernel of some homomorphism φ.

Only **normal** subgroups can be obtained as $\ker \varphi$.

Definition $H \subset G$ **normal** if $\forall g \in G, h \in H$ we have $ghg^{-1} \in H$

Claim $\ker \varphi \subset G$ is a normal subgroup

proof take $h \in \ker \varphi$, $g \in G \Rightarrow$

$\Rightarrow \varphi(ghg^{-1}) = \varphi(g) \cdot \varphi(h) \cdot \varphi(g)^{-1} =$

$$= \varphi(g) \cdot 1 \cdot \varphi(g)^{-1} = \underline{1}$$

So, $ghg^{-1} \in \ker \varphi$ ▨

Examples

If G a commutative then __any__ subgroup of G is normal.

For $G = S_3$ we have 6 subgroups:

✓ = normal
× = not normal

$\{1\}$ ✓

$\{1, (12)\}, \{1, (13)\}, \{1, (23)\}$ ×

$\{1, (123), (132)\} = A_3$ ✓ $\Rightarrow$ S_3 has 3 normal subgroups

S_3 ✓

Note: $A_3 \trianglelefteq$ indeed can be realized

as $\ker \varphi$, where $\varphi : S_3 \to \{\pm 1\}$

$$\begin{array}{ccc} S_3 & \to & \{\pm 1\} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma & \mapsto & (-1)^{\ell(\sigma)} \end{array}$$

Idea (later will become thm):

if $\varphi : G \to G'$ and $H := \ker \varphi$, then

 quotient has a group structure

1) $G' \underset{\text{iso. of groups}}{\simeq} G/H := \{ gH \mid g \in G \}$

2) φ identifies with
$$\begin{array}{ccc} G & \to & G/H \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ g & \mapsto & gH \end{array}$$

before proving this result let's discuss

a notion of a coset

Cosets

Let $H \subset G$ <u>arbitrary</u> subgroup; $g \in G$

then $gH := \{ gh \mid h \in H \}$ is called

<u>left</u> coset.

Note that H itself is a left coset

for $g = 1$.

We can define an equivalence relation

on G: $a \sim b$ if $a^{-1} b \in H$

$\Updownarrow$

$b \in aH$

Note that $\sim$ has the following properties:

(i) if $a \sim b$, $b \sim c \Rightarrow a \sim c$ (transitive)

(ii) if $a \sim b \Rightarrow b \sim a$ (symmetric)

(iii) $a \sim a$ (reflexive)

proof.

Start with (i): $a \sim b \Leftrightarrow a^{-1}b \in H$
$b \sim c \Rightarrow b^{-1}c \in H$

$$(a^{-1}b) \cdot (b^{-1}c) \in H$$

$$a \sim c \Leftarrow a^{-1}c$$

Now part (ii): $a \sim b \Rightarrow a^{-1}b \in H$

$$\Downarrow$$

$$b^{-1}a \in H$$

$$\Downarrow$$

$$b \sim a$$

Part (iii): $a \sim a$ because $a^{-1} \cdot a = 1 \in H$

__Remark:__ if relation $\sim$ on some set X satisfies (i), (ii), (iii), then it is called __equivalence relation.__

__Exercise:__ equivalence relation on X is the same as partition of X into a disjoint union of subsets

Going back to the case $X = G$ and $\sim$ given by $a \sim b \iff b \in aH$ we see that:

- equivalence classes for $\sim$ are <u>cosets</u> aH
- we have $G = \bigsqcup aH$

<u>Example</u> $G = S_3$, $H = \{1, (12)\}$

$$\Downarrow \qquad \qquad (12) = y$$

have <u>three</u> left cosets:

$$H = \underbrace{\{1, (12)\}}_{}; \quad (123)H = \underbrace{\{(123), (13)\}}_{}; \quad (132)H = \underbrace{\{(132), (23)\}}_{}$$

indeed partition $G = S_3$

We define:

$$G/H := \{ gH \mid g \in G \}$$

set of <u>equivalence classes</u> for $\sim$

<u>Note:</u> G ← was a group but

G/H ← may <u>not</u> have a group structure, it's just a <u>set</u>.

It's tempting to say that G/H is a group with a product given by: $(aH) \cdot (bH) = (abH)$ $(*)$

The problem is that this procedure is ~~NOT~~ **NOT** well-defined in general.

Example: $G = S_3$, $H = \{1, (12)\}$

$$\begin{array}{ccc} (123)H \cdot (132)H & = & H \\ \| & & \cancel{\|} \\ (13)H \cdot (132)H & = & (23)H \end{array}$$

problem!

Claim: if $H \subset G$ is normal then G/H has a group structure given by the formula $(*)$

proof.

Need to check that the product (*) is well-defined.

In other words need to check that

if $a_1 \sim b_1$, $a_2 \sim b_2 \Rightarrow a_1 a_2 \sim b_1 b_2$

Indeed $b_1 = a_1 h_1$, $b_2 = a_2 h_2$

for some $h_1, h_2 \in H$

Then $b_1 b_2 = a_1 h_1 \cdot a_2 h_2 =$

$$= a_1 a_2 \cdot \overbrace{\underbrace{(a_2^{-1} h_1 a_2)}_{\in H} \cdot h_2}^{\text{in } H \checkmark}$$

use that $H \trianglelefteq$ normal

Upshot: $H \subset G$

subgroup

Can form G/H ← set

becomes a group if $H \subset G$

normal

Example $G = S_3$, $H = A_3 = \{1, (123), (132)\}$

Then: G/H ← consists of two elements

$= \{H, (12)H\}$

$\cong$ ← iso of groups

$\{\pm 1\}$

$$H \mapsto 1, \quad (12)H \mapsto -1$$

If $\underset{\text{normal}}{H} \subset G$, then by definitions

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G/H \\ \in & & \in \\ g & \longmapsto & gH \end{array}$$

surjective homom. with kernel $= H$

Exercise: if $\varphi : G \to G'$ surjective homomorphism

$\Downarrow$

$G' \simeq G/H$, after this identification

φ becomes $G \to G/H$ as above

Upshot: surjective homomorphisms $G \to G'$

$\Updownarrow$

normal subgroups $H \subset G$