Lecture 6

Last time: homomorphisms and isomorphisms

$\varphi : G \to G'$ homomorphism if

$$\varphi(ab) = \varphi(a)\varphi(b)$$

φ isomorphism if it's bijective

Had examples:

"trivial"

$G \to G'$, $g \mapsto e_{G'}$

$G \to G$; $g \mapsto g$ "identity"

$\mathbb{Z} \xrightarrow{\cdot a} \mathbb{Z}$; $m \mapsto am$ (a any integer)

when iso?

$\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, $m \mapsto m \bmod n$

$(\mathbb{R},+) \to (\mathbb{R}^\times, \cdot)$, $t \mapsto 2^t$

Basic properties of homomorphisms

Proposition Let $\varphi: G \to G'$ a homomorphism

(a) If $a_1, \dots, a_k \in G$ then $\varphi(a_1 \dots a_k) = \varphi(a_1) \dots \varphi(a_k)$

(b) $\varphi(e_G) = e_{G'}$

(c) $\varphi(a^{-1}) = \varphi(a)^{-1}$

proof (a) exercise (use induction!
+
$a_1 \dots a_n = (a_1 \dots a_{n-1}) \cdot (a_n)$)

Sketch: $\varphi(a_1 \dots a_n) = \varphi((a_1 \dots a_{n-1}) \cdot (a_n)) = \varphi(a_1 \dots a_{n-1}) \cdot \varphi(a_n) =$

$\overset{\text{induction}}{=} (\varphi(a_1) \cdots \varphi(a_n)) \cdot (\varphi(a_n)) = \varphi(a_1) \dots \varphi(a_n)$

(b) Want to check that $\varphi(e_G) = e_{G'}$

Know: $e_G \cdot e_G = e_G$

$$\Downarrow$$

$$\varphi(e_G \cdot e_G) = \varphi(e_G)$$

$$\|$$

$$\cancel{\varphi(e_G)} \cdot \varphi(e_G) = \cancel{\varphi(e_G)} \quad \text{(cancellation law)}$$

$$\Downarrow$$

$$\varphi(e_G) = e_{G'} \quad \checkmark$$

(c) Want: $\varphi(a^{-1}) = \varphi(a)^{-1}$?

~~[illegible]~~ Need to check:

$$\varphi(a^{-1}) \cdot \varphi(a) = e_{G'} = \varphi(a) \cdot \varphi(a^{-1})$$

$$\varphi(a^{-1}) \cdot \varphi(a) = \varphi(a^{-1} \cdot a) = \varphi(e_G) = e_{G'} \quad \checkmark$$

$$\varphi(a) \cdot \varphi(a^{-1}) = \varphi(a \cdot a^{-1}) = \varphi(e_G) = e_{G'} \quad \checkmark$$

Isomorphisms

Recall: $\varphi : G \to G'$ ⊸ isomorphism if

- φ ⊸ homomorphism
- φ ⊸ bijective

Lemma if $\varphi : G \to G'$ ⊸ isomorphism, then $\varphi^{-1} : G' \to G$ ⊸ is an isomorphism

proof Need to check that:

$$\underbrace{\varphi^{-1}(xy)}_{\text{mult. in } G'} = \underbrace{\varphi^{-1}(x)\,\varphi^{-1}(y)}_{\text{mult. in } G} \quad \forall x, y \in G'$$

Set $a := \varphi^{-1}(x)$; $b := \varphi^{-1}(y)$; $c := \varphi^{-1}(xy)$

$c = ab$ ← want to check this

φ a bijective, so enough to check:

$$\varphi(c) = \varphi(ab)$$

Indeed: $\varphi(ab) = \varphi(a)\varphi(b) = xy = \varphi(c)$ ✓

Two groups G, G' are <u>isomorphic</u> if $\exists\, \varphi : G \xrightarrow{\sim} G'$. We will write $G \simeq G'$.

a <u>isomorphism</u>

Warning: isomorphism is <u>NOT</u> unique in general!

Examples (of isomorphic and not isomorphic groups)

$$(\mathbb{Z}^\times, \cdot) = \{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}$$

$$1 \longmapsto 0$$

$$-1 \longmapsto 1$$

$$\begin{array}{c} S_3 \\ \cup \\ \langle (123) \rangle \simeq \mathbb{Z}/3\mathbb{Z} \\ \rotatebox{90}{$\in$} \end{array}$$

$$1 \longmapsto 0$$

$$(123) \longmapsto 1$$

$$(132) \longmapsto 2$$

NOT isomorphic

note that both contain 4 elements

$$(\mathbb{Z}/12\mathbb{Z})^\times \not\simeq \mathbb{Z}/4\mathbb{Z}$$

every element has order ≤ 2

has element of order 4

More examples

$$\{1, x, \ldots, x^{n-1}\}$$
$$\parallel$$
$$x \in G, \ \mathrm{ord}(x) = n \Rightarrow \langle x \rangle_G \simeq \mathbb{Z}/n\mathbb{Z}$$
$$\psi$$
$$x^k \longmapsto k$$

$$x \in G, \ \mathrm{ord}(x) = \infty \Rightarrow \langle x \rangle_G \simeq \mathbb{Z}$$

We see that two isomorphic groups are "the same" and we do not want to distinguish them.

- The groups isomorphic to a given group G from what is called <u>isomorphism class</u> of G.

<u>Lemma</u>: if $G_1 \simeq G_2$; $G_2 \simeq G_3 \Rightarrow G_1 \simeq G_3$

in other words, any two groups in the isomorphism class are isomorphic

Lemma above follows from the following general claim.

Claim if $\underbrace{\varphi : G \to H \, ; \, \Psi : H \to S}_{\text{homomorphisms of groups}}$

then $\Psi \circ \varphi$ is a homomorphism

is it clear how Lemma follows from Claim?

Proof $(\Psi \circ \varphi)(ab) = \Psi(\varphi(ab)) =$

$= \Psi(\varphi(a)\varphi(b)) = \Psi(\varphi(a))\Psi(\varphi(b)) =$

$= (\Psi \circ \varphi)(a) \cdot (\Psi \circ \varphi)(b)$ ✓

Very ~~[illegible]~~ complicated problem classify all groups

too hard!

i.e., describe their isomorphism classes

Will see every group of order p is $\simeq \mathbb{Z}/p\mathbb{Z}$

Question: can you describe groups of order 4?

More examples of isomorphisms

identity iso.

Clearly $G \xrightarrow{\sim} G$ via $g \mapsto g$

Interestingly, in general, there are <u>many ways</u> to identify G with itself.

An isomorphism $G \xrightarrow{\sim} G$ is called <u>automorphism</u>.

Example of interesting automorphism:

$$S_3 = \langle \underset{(123)}{\underset{\parallel}{x}}, \underset{(12)}{\underset{\parallel}{y}} \rangle \Big/ \begin{matrix} x^3 = 1 \\ y^2 = 1 \\ yx = x^2 y \end{matrix} = \{1, x, x^2, y, xy, x^2 y\}$$

Exercise: $\begin{aligned} x &\overset{\varphi}{\longmapsto} x^2 \\ y &\longmapsto y \end{aligned}$ extends to the automorphism of S_3.

Explicitly, we have

$$\begin{aligned} 1 &\longmapsto 1 \\ x &\longmapsto x^2 \\ x^2 &\longmapsto x \\ y &\longmapsto y \\ xy &\longmapsto x^2 y \\ x^2 y &\longmapsto xy \end{aligned}$$

To solve this exercise need to check that relations are preserved

$$x^3 = 1 \xrightarrow{\text{apply } \varphi} (x^2)^3 = x^6 = 1 \ \checkmark$$

$$y^2 = 1 \xrightarrow{\text{apply } \varphi} y^2 = 1 \ \checkmark$$

$$yx = x^2 y \xrightarrow{\text{apply } \varphi} y x^2 \overset{?}{=} x^4 y$$

Want: $y x^2 = x^4 y \overset{x^3=1}{\iff} y x^2 = x y \overset{y\cdot}{\iff} x^2 = y x y$

$\Updownarrow \cdot y$

$x^2 y = y x$ ✓

In general: if $G = \langle x_1, \dots, x_k \rangle / \text{relations}$

then to define homomorphism:

$\varphi: G \to S$ ← some other group

<u>Need to:</u> 1) define φ on generators

$x_1 \overset{\varphi}{\longmapsto} y_1$

$\vdots$

$x_k \mapsto y_k$

2) check that relations are preserved:

if $x_1^{q_1} \cdots x_k^{q_k} = 1 \implies y_1^{q_1} \cdots y_k^{q_k} = 1$

equality in G — equality in S

The automorphism $\varphi: S_3 \to S_3$

that we constructed is a particular case ~~construction~~ of very general construction:

G a group, $g \in G$ any element

Exercise: $G \xrightarrow{\varphi_g} G$, $h \overset{\varphi_g}{\longmapsto} ghg^{-1}$ (conjugate of h by g) is automorphism of G.

Example: $\varphi: S_3 \to S_3$ corresponds to $g = y = (12)$

$y \mapsto y \cdot y \cdot y^{-1} = y$

(indeed $x = (123) \mapsto (12)(123)(12) = (132) = x^2$)