Lecture 17

Last time

- Cauchy's thm: G grp finite, $p \mid |G|$

$\Downarrow$

G contains an element of order p

- Proved Cauchy's thm for p-groups, commutative

- Formulated first Sylow's thm:

G grp finite, $|G| = p^k \cdot n$ $\quad (\gcd(p,n) = 1)$

$\Downarrow$

$\exists H \subset G$, $|H| = p^k$

Sylow's p-group

- Sylow's thm $\Rightarrow$ Cauchy's thm

First goal for today ← prove Sylow's first thm.

<u>Examples</u> $G = S_3$, $p = 3 \Rightarrow |H| = 3$

$H = \langle (1\,2\,3) \rangle$

$G = S_4$, $p = 2 \Rightarrow |H| = 8$, $H = D_8 \hookrightarrow S_4$ ✓

<u>proof</u>

<u>Strong</u> induction on $|G|$

Two cases: $p \mid |Z|$ / $p \nmid |Z|$ (Z ← center of G)

① $p \mid |Z| \Rightarrow Z \supset N$

Cauchy's thm. for Z (commutative)

Subgroup of order p

Set $\bar{G} := G/N \Rightarrow |\bar{G}| = p^{k-1} \cdot n$

Induction $\Rightarrow \exists \bar{H} \subset \bar{G}$

order p^{k-1}

Consider

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \bar{G} \\ \cup & & \cup \\ H & \longrightarrow & \bar{H} \\ \| & & \\ \pi^{-1}(\bar{H}) & & \end{array}$$

kernel $\qquad$ image

$$\underset{p}{N} \subset H \twoheadrightarrow \underset{p^{k-1}}{\bar{H}}$$

subgroup of G, $|H| = p \cdot |\bar{H}| =$

$= p^k$ ✓

② $p \nmid |Z|$

$$G = Z \sqcup O_{x_1} \sqcup \dots \sqcup O_{x_r}$$

conj. classes of non-central elements x_i

$$|G| = |Z| + \sum_{i=1}^{r} [G : G_{x_i}]$$

$p \mid |G|$, $p \nmid |Z| \Rightarrow \exists i$ s.t. $p \nmid [G : G_{x_i}]$

Then $|G_{x_i}| = p^k \cdot ?$

$G_{x_i} \subset G$

proper subgroup of G
(x_i non-central)

induction hypothesis $\Rightarrow \exists H \subset G_{x_i}$

$|H| = p^k$ ✓

So, we know that G contains

$H \subset G$

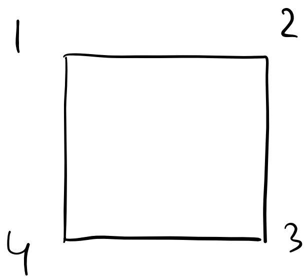
Sylow p-subgroup.

But H is $\underline{\underline{\text{not}}}$ unique!

Ex: $G = S_4$, $p = 2$ $\Rightarrow$ $D_8 \hookrightarrow S_4$

$$D_8 = \langle \underset{s}{(24)}, \underset{sr}{(14)(23)} \rangle = H$$

works



But also $\langle (12), \overset{4\,1\;2\,3}{(13)(24)} \rangle = H'$ works

(see midterm)

They are $\underline{\underline{\text{different}}}$ Sylow p-subgroups.

Claim: H and H' are <u>conjugate</u>

i.e. $\exists g \in S_4$ s.t. $gHg^{-1} = H'$

Indeed: take $g = (134)$ then:

$$(134)\cdot(24)\cdot(134)^{-1} = (12) \quad ✓$$

$$(134)\cdot(14)(23)\cdot(134)^{-1} = (13)(24) \quad ✓$$

(in particular $H' \simeq H$)

It turns out that this is <u>always</u> the case.

Sylow p-subgroup

Thm (second Sylow's thm), fix $H \subset G$

a) if $H' \subset G$ another Sylow's subgroup $\Rightarrow$
$\Rightarrow \exists g \in G$ s.t. $H' = gHg^{-1}$

b) if $K \subset G$ <u>any</u> p-subgroup then

$\exists g \in G$ s.t. $K \subset gHg^{-1}$

<u>proof</u>: enough to prove b)

$$\left(H' \subset gHg^{-1} \Rightarrow H' = gHg^{-1} \right)$$
$$|H| = |H'|$$

To prove b), consider:

$$G \curvearrowright G/H =: X$$
$$g \cdot aH = gaH$$

$$K \subset G \Rightarrow K \curvearrowright X$$
subgroup

$$|X| = \frac{|G|}{|H|} = n \Rightarrow p \nmid |X|$$

Claim: $\exists x \in X$ s.t. $K_x = K$

This follows from the general lemma:

Lemma

If $K \curvearrowright X$, K a p-group, $p \nmid |X|$

$\Downarrow$

$\exists x \in X$ s.t. $K_x = K$

proof. $|X| = \sum' [K : K_{x_i}]$, $p \nmid |X|$

$\Downarrow$

$\exists x_i = x$ s.t. $p \nmid [K : K_x] = \dfrac{|K|}{|K_x|} = \dfrac{p^e}{|K_x|}$

$\Downarrow$

$|K_x| = p^e$ and $K_x = K$ ✓

So, we can find $x \in X$ s.t. $K_x = K$

$\Downarrow$

$K \subset G_x$

Recall, $x = gH$ for some $g \in G$

$\Downarrow$

$G_x = gHg^{-1}$

We conclude that $K \subset gHg^{-1}$ ✓

<u>Upshot</u>: G a finite group, $|G| = p^k \cdot n$

- $\exists H \subset G$ $|H| = p$
- all of these H are <u>isomorphic</u> and are of the form gHg^{-1} for some g
- if $K \subset G$ a <u>any</u> p-subgroup $\Rightarrow K \subset gHg^{-1}$

Final question: how many of these H do we have?

Answer:

Thm (third Sylow's thm), $|G| = p^k \cdot n$

Let s ← number of Sylow p-subgroups

Then:

- $s \mid m$
- $s \equiv 1 \mod p$

Example $G = S_3$, $p = 3$ $\Rightarrow$ $H = \langle (123) \rangle$

only one $\quad s = 1$

$|G| = 2^2 \cdot \underset{}{3}$ $\quad$ " m

$$G = S_4 \Rightarrow H = \begin{cases} \langle (24), (14)(23) \rangle \\ \langle (12), (13)(24) \rangle \\ \langle (14), (13)(24) \rangle \end{cases}$$

← $s = 3 \mid 3$

$3 \equiv 1 \mod 2$

proof.

Consider $G \curvearrowright X$ ← set of all Sylow p-subgroups

$$g \cdot H = gHg^{-1}$$

We know (by second Sylow thm) that

$G \curvearrowright X$ — **transitive**

Fix $H \in X$, $G_H = \{ g \in G \mid gHg^{-1} = H \}$

$$\begin{array}{c} G_H \\ \cup \\ H \end{array}$$

$$s = |X| = \frac{|G|}{|G_H|} \;\Big|\; \frac{|G|}{|H|} = m$$

$s \mid m$ ✓

Now consider $H \curvearrowright X$ (restriction of $G \curvearrowright X$ to H)

$H \in X$ is fixed by H.

To show that $s \equiv 1 \mod p$ enough to check that <u>no</u> other H' is fixed by H (use counting formula)

If H fixes H' $\Rightarrow$ $\forall h \in H$, $hH'h^{-1} = H'$

so $H \subset G_{H'}$.

Sylow p-subgroup of $G_{H'}$

But H' also lives in $G_{H'}$.

Second Sylow thm applied to $G_{H'}$

implies that $\exists\, g \in G_{H'}$ s.t. $gH'g^{-1} = H$

But $gH'g^{-1} = H'$ (see the definition of $G_{H'}$)

$\Downarrow$

$H = H'$ ✓