Lecture 15

Last time

- $G \curvearrowright X \rightsquigarrow X = \bigsqcup \mathcal{O}$

partition into G-orbits

(x, y are in the same orbit if $y = gx$ for some $g \in G$)

conjugation

Ex: $S_3 \curvearrowright S_3 = \{1\} \sqcup \{(12),(23),(13)\} \sqcup \{(123),(132)\}$

conj. classes

- $G \curvearrowright \mathcal{O}$ ← action is **transitive**

($\forall x, y \in \mathcal{O}\ \exists g$ s.t. $y = gx$)

- if $H \subset G$ (subgroup) $\Rightarrow G \curvearrowright G/H$

transitive

How the action $G \curvearrowright G/H$ works:

$$g \cdot (aH) = gaH$$

with $g \in G$ and $aH \in G/H$.

<u>Example:</u> $G = S_3$, $H = \{1, (12)\}$

$$G/H = \{ \underbrace{\{1, (12)\}}_{C_1}, \underbrace{\{(23), (132)\}}_{C_2}, \underbrace{\{(13), (123)\}}_{C_3} \}$$

$$(12) \curvearrowright \{C_1, C_2, C_3\}$$

to see that: $(12) \cdot \{1, (12)\} = \{(12), 1\} = C_1$

$$(12) \cdot \{(23), (132)\} = \{(123), (13)\} = C_3$$

$$(12) \cdot \{(13), (123)\} = \{(132), (23)\} = C_2$$

$$(123) \curvearrowright \{C_1, C_2, C_3\}$$

$$(123)\cdot C_1 = \{(123), (13)\} = C_3$$

$$(123)\cdot C_2 = C_1$$ ← Check!

$$(123)\cdot C_3 = C_2$$

How $(23) = (12)(123)$ acts?

$$C_1 \xmapsto{(123)} C_3 \xmapsto{(12)} C_2$$

$$C_2 \xmapsto{(123)} C_1 \xmapsto{(12)} C_1$$

$$C_3 \xmapsto{(123)} C_2 \xmapsto{(12)} C_3$$

$$\Rightarrow (23) \curvearrowright \{C_1, C_2, C_3\}$$

In general, why $G \curvearrowright G/H$

$g \cdot (aH) = gaH$

is indeed an action:

1) $1 \cdot (aH) = (1 \cdot a)H = aH$ ✓

2) $g_1 \cdot (g_2 \cdot (aH)) = g_1 g_2 a H = g_1 g_2 \cdot (aH)$ ✓

Note: $H \subset G$ ← need not be normal!

Claim: action $G \curvearrowright G/H$ is transitive (only one orbit)

prf every element of G/H is gH for some $g \in G$, now $gH = g \cdot H \Rightarrow gH \sim H$ ✓

It turns out that <u>any</u> orbit $\mathcal{O}$ for

$G \curvearrowright X$ can be identified with G/H

for appropriate $H \subset G$.

Namely, if $x \in \mathcal{O}$, we can define:

$$G_x := \{ g \in G \mid g \cdot x = x \} \subset G$$

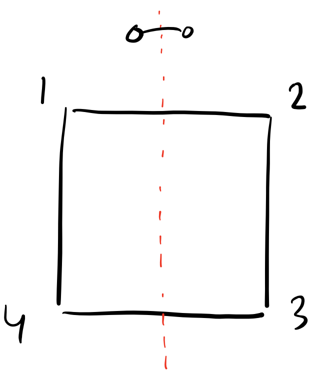
↖ <u>stabilizer</u> of G

Will see that $\mathcal{O} \simeq G/G_x$

as sets with G-action

Examples of stabilizers

$S_4 \curvearrowright \{1,2,3,4\}$, stabilizer of $\{4\}$ is $S_3 \subset S_4$

$D_4 \curvearrowright$ 

$\rightsquigarrow$

$D_4 \curvearrowright \{[12],[14],[23],[34]\}$

Stabilizer of $[12]$ is $\{1, (12)\}$, $(12) = rs$

For $G \curvearrowright G/H$ stabilizer of $H \in G/H$ is $H \subset G$ $(gH = H \Rightarrow g \in H)$

So, can "read off" H from $G \curvearrowright X = G/H$

Claim $G_x \subset G$ subgroup

prf. $1 \in G_x$ as $1 \cdot x = x$

if $g_1, g_2 \in G_x \Rightarrow (g_1 g_2) \cdot x = g_1 \cdot (g_2 \cdot x) =$

$= g_1 \cdot x = x$ ✓

Propos. if $\mathcal{O} \subset X$ orbit for $G \curvearrowright X$

and $x \in \mathcal{O} \Rightarrow$ there exists an identification:

identification of sets with G-action →

$$g \in G \quad\longrightarrow\quad G/G_x \xrightarrow[\sim]{\pi} \mathcal{O}$$

$$G/G_x \ni g G_x \longmapsto g \cdot x \in \mathcal{O}$$

← similar to what we discussed before (structure of surjective morphisms)

<u>Proof</u> 1) $gG_x \longmapsto g \cdot x$ is well-defined

take $a \in G_x$, $(ga)\cdot x = g\cdot(a\cdot x) = g \cdot x$ ✓

2) π surjective: every element of $\mathcal{O}$ is of the form $g\cdot x$ for some $g \in G$

(use that $\mathcal{O}$ orbit)

3) π injective: if

$$\begin{aligned} g_1 G_x &\longmapsto g_1 \cdot x \\ & \qquad\ \shortparallel \\ g_2 G_x &\longmapsto g_2 \cdot x \end{aligned}$$

$\Downarrow$

$$(g_2^{-1} g_1)\cdot x = g_2^{-1}\cdot(\underbrace{g_1\cdot x}_{g_2 \cdot x}) = x$$

$\Downarrow$

$$g_2^{-1} g_1 \in G_x \Rightarrow g_2 G_x = g_2 \cdot (g_2^{-1} g_1) G_x =$$

$$= g_1 G_x \quad ✓$$

Applications

Counting formula 1

$G \curvearrowright X$ (finite)

Let $\mathcal{O} \subset X$ (orbit), $x \in \mathcal{O}$

Then: $|G| = |G_x| \cdot |\mathcal{O}|$; $|\mathcal{O}| = [G : G_x]$

proof. $\mathcal{O} \simeq G/G_x \Rightarrow |\mathcal{O}| = |G/G_x| = \frac{|G|}{|G_x|}$ ✓

Counting formula 2

$G \curvearrowright X = \mathcal{O}_1 \sqcup \ldots \sqcup \mathcal{O}_k$ (orbits), $x_i \in \mathcal{O}_i$

$\Downarrow$

$|X| = |\mathcal{O}_1| + \ldots + |\mathcal{O}_k| = [G : G_{x_1}] + \ldots + [G : G_{x_k}]$

Example: $S_3 \curvearrowright S_3$ (conjugation)

$$S_3 = \{1\} \sqcup \{(12),(23),(13)\} \sqcup \{(123),(132)\}$$

$$\{1\} = O_1, \quad \{(12),(23),(13)\} = O_2, \quad \{(123),(132)\} = O_3$$

$$G_1 = S_3; \quad G_{(12)} = \{1,(12)\} = \langle (12) \rangle, \quad G_{(123)} = \langle (123) \rangle$$

$$|S_3| = 6 = \frac{6}{6} + \frac{6}{2} + \frac{6}{3}$$

Application (see Prop. 7.3.1 in Artin's book)

G – finite group of order p^k $\left(\begin{array}{l} p\text{-prime} \\ k \in \mathbb{Z}_{\geq 1} \end{array} \right)$

such groups are called "p-groups"

Then $Z \subset G$ center is nontrivial

Proof. $G \curvearrowright G$ (conjugation)

conj. classes = orbits

$$G = O_1 \sqcup \dots \sqcup O_n$$

element $g \in G$ lives in Z iff $\{g\} = O_i$ for some i.

If $|O| > 1 \Rightarrow |O|$ ← divisor of p^k

$$\Downarrow$$

$$p \mid |O|$$

We see that:

$$|G| = |Z| + p \cdot ?$$

$$\Downarrow$$

$p \mid |Z| \Rightarrow Z$ contains at <u>least</u> p elements.

so Z is nontrivial!