Lecture 14

Last time: – $\overset{\text{group}}{G} \underset{\text{acts on}}{\curvearrowright} \overset{\text{set}}{X}$ if

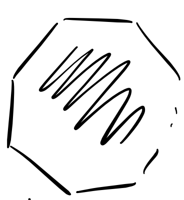
we are given a map
$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longmapsto g \cdot x \end{aligned}$$
s.t.

1) $1 \cdot x = x \quad \forall x \in X$

2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x \quad \forall g_1, g_2 \in G$

– Examples: $S_n \curvearrowright \{1, 2, \dots, n\}$

$$\sigma \cdot i = \sigma(i)$$

– $D_{2n} \curvearrowright$  $= X$, $f \cdot x = f(x)$

More generally: for any set X,

$$S_X \curvearrowright X$$

$$\sigma \cdot x = \sigma(x)$$

<u>proof</u> $\quad id \cdot x = id(x) = x \quad \checkmark$

$$\sigma_1 \cdot (\sigma_2 \cdot x) = \sigma_1(\sigma_2(x)) = (\sigma_1 \circ \sigma_2)(x) \checkmark$$

<u>More examples of actions</u>

$$S_4 \curvearrowright \{1,2,3,4\}$$

but it also acts on

$$S_4 \curvearrowright \left\{ \begin{array}{c} \{1,2\} \sqcup \{3,4\} \\ \{1,3\} \sqcup \{2,4\} \\ \{1,4\} \sqcup \{2,3\} \end{array} \right\}$$

For example: $(123)\cdot(\{12\}\sqcup\{34\}) = (\{14\}\sqcup\{23\})$

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$(123)\cdot(\{14\}\sqcup\{2,3\}) = (\{13\}\sqcup\{24\})$

Remember that we used this to construct

homomorphism $S_4 \to S_3$

homomorphism

Claim $G \curvearrowright X \iff G \to S_X$

action of G on X is the same as a homomorphism $G \to S_X$

prf. if $G \curvearrowright X \rightsquigarrow G \xrightarrow{\varphi} S_X$

$g \mapsto (x \mapsto gx)$

φ_σ is a homomorphism:

$$\varphi(g_1 g_2) \overset{?}{=} \varphi(g_1) \circ \varphi(g_2)$$

$$\Updownarrow$$

$$\varphi(g_1 g_2)(x) = \varphi(g_1)(\varphi(g_2)(x)) \quad \forall x \in X$$

$$(g_1 g_2) \cdot x \overset{\checkmark}{=} g_1 \cdot (g_2 \cdot x)$$

In the opposite direction:

if $\underbrace{\varphi : G \to S_X}_{\text{homomorphism}} \rightsquigarrow G \curvearrowright X$

$$g \cdot x = \varphi(g)(x)$$

check that this formula indeed defines an action

More examples of action

$$G \hookrightarrow S_G \rightsquigarrow G \curvearrowright G$$

$$\begin{array}{ccc} \Psi & & \varphi \\ g \longmapsto \left(h \overset{m_g}{\longmapsto} gh\right) & & g \cdot h = gh \end{array}$$

Ex: $S_3 \curvearrowright S_3$

$$\begin{array}{ll} \Psi & m_{(12)} \\ (12) & 1 \longmapsto (12) \\ & (12) \longmapsto 1 \\ & (23) \longmapsto (123) \\ & \vdots \end{array}$$

$$G \hookrightarrow S_G \rightsquigarrow G \curvearrowright G$$

$$\begin{array}{ccc} \Psi & & \\ g \longmapsto \left(h \longmapsto hg^{-1}\right) & & g \cdot h = hg^{-1} \end{array}$$

let's check this is indeed an action

$$g_1 \cdot (g_2 \cdot h) = g_1 \cdot (h g_2^{-1}) = h g_2^{-1} g_1^{-1} =$$

$$= h (g_1 g_2)^{-1} = (g_1 g_2) \cdot h$$

See that inverse is <u>crucial</u>

$$G \longrightarrow \mathrm{Aut}(G) \subset S_G \rightsquigarrow G \curvearrowright G$$

$$g \longmapsto (h \mapsto g h g^{-1}) \qquad g \cdot h = g h g^{-1}$$

If $H \subset G$ and $G \curvearrowright X \Rightarrow H \curvearrowright X$

In particular, $H \curvearrowright G$, $h \cdot g = \begin{cases} hg \\ gh^{-1} \\ hgh^{-1} \end{cases}$

Orbits and stabilizers

Consider the following example:

$$S_3 \curvearrowright S_3$$

conjugation

$\{1\}$ ← preserved by the action

$$(12) \xrightarrow{(123)} (23) \xrightarrow{(12)} (13) \quad \leftarrow \{(12), (23), (13)\}$$

another conjugacy class

$$\{(123) \xrightarrow{(12)} (132)\}$$

$$S_3 = \{1\} \sqcup \{(12), (23), (13)\} \sqcup \{(123), (132)\}$$

"orbits" for $S_3 \curvearrowright S_3$

In general:

$G \curvearrowright X$ ← partitions X into orbits

Namely: we say that $x, y \in X$ are equivalent if $\exists g \in G$ s.t. $y = gx$ $(x \sim y)$

Claim: $\sim$ ← equivalence relation

$x \sim x \quad (x = 1 \cdot x)$

$x \sim y \implies y \sim x \quad (\text{if } y = g \cdot x \implies x = g^{-1} \cdot y)$

$x \sim y \sim z \implies y = g_1 x,\ z = g_2 y$

so $z = g_2 \cdot (g_1 x) = (g_2 g_1) \cdot x$

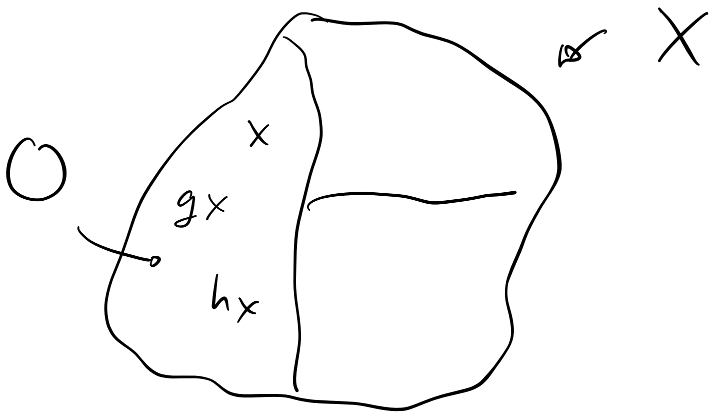
$\Downarrow$

$x \sim z$

Get $X = \bigsqcup \mathcal{O}$

equivalence classes for $\sim$

to be called <u>orbits</u> for the action



<u>Example</u>

$H \curvearrowright G$, $h \cdot g = hg$

$\Downarrow$

H-orbits are <u>right cosets</u> Hg

$S_3 \curvearrowright S_3$

$$\underset{\{1\}}{\overset{S_3}{\curvearrowright}} \sqcup \underset{\{(12),(23),(13)\}}{\overset{S_3}{\curvearrowright}} \sqcup \underset{\{(123),(132)\}}{\overset{S_3}{\curvearrowright}}$$

We see that many objects we introduced before are particular cases of this general structure.

<u>Note</u> G acts on every orbit $O \subset X$

"identifies" all elements in this orbit

If $G \curvearrowright X$ with <u>one</u> orbit then this action is called <u>transitive</u>

Example: $S_n \curvearrowright \{1, 2, \ldots, n\}$

is transitive

Example $G \curvearrowright G/H$

$$g \cdot (aH) = gaH$$

note that H need not be normal!

1) $1 \cdot (aH) = (1 \cdot a)H = aH$ ✓

2) $g_1 \cdot (g_2 \cdot (aH)) =$

$= g_1 g_2 aH = (g_1 g_2) \cdot aH$ ✓

Action $G \curvearrowright G/H$ $\rightsquigarrow$ transitive

It turns out that <u>any</u> orbit $\mathcal{O}$ for $G \curvearrowright X$ can be identified with G/H for appropriate $H \subset G$

Namely, if $x \in \mathcal{O}$, we can define $G_x := \{ g \in G \mid g \cdot x = x \}$

<u>stabilizer</u> of x

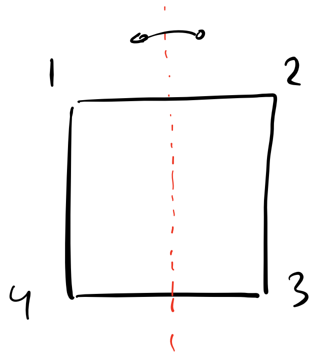
We will see that $\mathcal{O} \simeq G/G_x$

as sets with G-action

<u>Examples of stabilizers</u>

$$S_4 \curvearrowright \{1, 2, 3, 4\}$$

Stabilizer of $\{4\}$ is $S_3 < S_4$

$D_4 \circlearrowright$ 

$D_4 \circlearrowright \{[12], [14], [23], [34]\}$

stabilizer of $[12]$ is $\{1, (12)\}$

For $G \circlearrowright G/H$ stabilizer of

$H \in G/H$ is $H < H$

So, can "read off" H from $G \circlearrowright X$

$$X = G/H$$

__Claim__ $G_x \subset G$ subgroup

__prf__ $1 \in G_x$ as $1 \cdot x = x$ ✓

if $g_1, g_2 \in G_x \Rightarrow (g_1 g_2) \cdot x =$

$= g_1 \cdot (g_2 \cdot x) = g_1 \cdot x = x$ ✓

__Propos.__ if $\mathcal{O} \subset X$ orbit for $G \curvearrowright X$

and $x \in \mathcal{O} \Rightarrow$ there exists an identification

$$\begin{array}{ccc} G/G_x & \xrightarrow[\simeq]{\pi} & \mathcal{O} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ gG_x & \longmapsto & g \cdot x \end{array}$$

identification of sets with G-action

will prove this next time