<u>Last time</u> $\mathcal{F}(S)$ $S = \{x_1, \dots, x_n\}$

- Defined $\overset{\swarrow}{\langle x_1, \dots, x_n \rangle} \Big/ \langle R \rangle_{\text{norm}}$

- Main thm: $\forall G$, we have

$$\mathcal{F}(S) \big/ \langle R \rangle_{\text{norm}} \xrightarrow{\psi} G$$

$$\Updownarrow$$

1) elements $y_i \in G$

2) s.t. $\forall\, x_{i_1} \cdots x_{i_k} \in R$, $y_{i_1} \cdots y_{i_k} \overset{\text{equality in } G}{=} 1$

- reduced thm to the following general result:

<u>Thm</u> Let $f : S \to S'$ ← group homom. with kernel K

Let $N \subset S$ ← normal <u>contained</u> in K

Then there is a unique map $S/N \xrightarrow{h} S'$

s.t.

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr] & S' \\ & S/N \arrow[u, "h"'] \end{tikzcd}$$

is commutative

it is given by $gN \mapsto f(g)$

proof. we proved that

there exists an iso.

$$\begin{array}{ccc} a: S/K & \xrightarrow{\sim} & \operatorname{im} f \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ gK & \longmapsto & f(g) \end{array}$$

$$\begin{tikzcd} S \arrow[r, two heads] \arrow[rr, bend left, "f"] \arrow[dr, two heads, "\varphi"'] & \operatorname{im} f \arrow[r, hook] & S' \\ & S/K \arrow[u, "a", "\sim"'] & \end{tikzcd}$$

In other words, f identifies with:

$$\begin{tikzcd} S \arrow[r, two heads, "\varphi"] \arrow[rr, bend left, "f"] & S/K \arrow[r, hook] & S' \\ g \arrow[r, mapsto] & gK \arrow[r, mapsto] & f(g) \end{tikzcd}$$

Using that $N \subset K$ we conclude that

φ factors through:

this is the desired $\underline{\underline{h}}$

$$\begin{tikzcd}
S \arrow[r, two heads] & S/N \arrow[r, two heads] \arrow[rr, bend left] & S/K \arrow[r, hook] & S' \\
g \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & gN \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & gK \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & f(g) \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

<u>Claim</u> $D_{2n} \simeq \langle r, s\rangle \big/ \langle r^n, s^2, (sr)^2\rangle_{\text{norm}}$

will often omit

<u>proof.</u> By our main thm, homomorphism

$$\underbrace{\langle r, s\rangle \big/ \langle r^n, s^2, (sr)^2\rangle}_{G} \xrightarrow[\varphi]{\substack{r \mapsto r \\ s \mapsto s}} D_{2n}$$

is well-defined

1) it is surjective ← r, s generate D_{2n}

2) remains to check that $|G| \leq 2n$

then ψ must be iso.

(we proved that $|D_{2n}| = 2n$)

Exercise: every element of G is of the form

r^k or sr^k $\Rightarrow |G| \leq 2n$

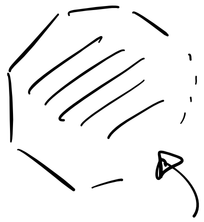
$(k = 0, 1, \dots, n-1)$

Recall D_{2n} ← group of symmetries

of  ← n-gon

S_n ← group of "permutations" of $\{1, \dots, n\}$

We see that $D_{2n} \overset{g}{\curvearrowright}$



"acts on"

meaning $\forall x \in P$, $g \in D_{2n}$

$$g(p) \in P$$

Similarly $$\begin{array}{c} S_n \\ \rotatebox{90}{$\in$} \\ \sigma \end{array} \curvearrowright \{1, 2, \dots, n\} \qquad \sigma \cdot i = \sigma(i)$$

Let's "axiomatize" notion of a group

G acting on a set X:

will see that this notion is <u>very</u> <u>very</u> useful

Definition $G \curvearrowright X$ is a map

two notations

$$G \times X \to X \quad \leftarrow \quad (g, x) \mapsto g \cdot x = gx$$

s.t.

1) $1 \cdot x = x \quad \forall x \in X$

2) $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$

Examples: $S_n \curvearrowright \{1, 2, \dots, n\}$

1) $1 \cdot i = 1(i) = i$ ✓

2) $\sigma_1 \cdot (\sigma_2 \cdot i) = \sigma_1(\sigma_2(i)) =$

$= (\sigma_1 \circ \sigma_2)(i)$

$S_4 \curvearrowright$ set of pairs $\left\{ \begin{matrix} \{1,2\} \sqcup \{3,4\} \\ \{1,3\} \sqcup \{2,4\} \\ \{1,4\} \cup \{2,3\} \end{matrix} \right\}$

$\forall$ set X, $\quad S_X \curvearrowright X$

$\sigma \cdot x = \sigma(x)$

Claim

$$G \curvearrowright X \iff G \xrightarrow[\uparrow]{\varphi} S_X$$

$$\underline{\text{homomorphism}}$$

prf. if $G \curvearrowright X \Rightarrow$ get

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & S_X \\ \rotatebox{90}{$\in$} & & \\ g & \longmapsto & (x \mapsto gx) \end{array}$$

$$\varphi(g_1 g_2) \overset{?}{=} \varphi(g_1) \circ \varphi(g_2)$$

$$\Updownarrow$$

$$\varphi(g_1 g_2)(x) = \varphi(g_1)\big(\varphi(g_2)(x)\big) \quad \forall x \in X$$

$$\uparrow \qquad\qquad\qquad \uparrow$$

$$(g_1 g_2) \cdot x \overset{\checkmark}{=} g_1 \cdot (g_2 \cdot x)$$

If $\varphi : G \to S_X \rightsquigarrow G \curvearrowright X$

$$g \cdot x = \varphi(g)(x)$$

$\nearrow$ check that satisfies action axioms

More examples of action:

action by "left multiplication"

$$\begin{array}{ccc} G & \longrightarrow & S_G \\ \rotatebox{90}{$\in$} & & \\ g & \longmapsto & \overset{m_g}{(h \mapsto gh)} \end{array} \quad \rightsquigarrow \quad G \curvearrowright G$$

$$g \cdot h = gh$$

$$\begin{array}{ccc} G & \longrightarrow & S_G \\ \rotatebox{90}{$\in$} & & \\ g & \longmapsto & (h \mapsto hg^{-1}) \end{array} \quad \rightsquigarrow \quad G \curvearrowright G$$

$$g \cdot h = hg^{-1}$$

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(G) \subset S_G \\ \rotatebox{90}{$\in$} & & \\ g & \longmapsto & \overset{c_g}{(h \mapsto ghg^{-1})} \end{array} \quad \rightsquigarrow \quad G \curvearrowright G$$

$$g \cdot h = ghg^{-1}$$

If $H \overset{\text{subgroup}}{\subset} G$, $G \curvearrowright X \Rightarrow H \curvearrowright X$

In particular, $H \curvearrowright G$, $h \cdot g = hg$

Orbits and stabilizers

$G \curvearrowright X$ ⇝ **partitions** X into **orbits**

Namely: we say that $x, y \in X$ are **equivalent** if $\exists g \in G$ s.t. $y = gx$ $(x \sim y)$

Claim: $\sim$ ⇝ equivalence relation:

$x = 1 \cdot x \Rightarrow x \sim x$

if $y = g \cdot x \Rightarrow g^{-1} \cdot y = x$ so $x \sim y \Leftrightarrow y \sim x$

if $x \sim y \sim z \Rightarrow z = gy,\ y = hx \Rightarrow$

$\Rightarrow z = g \cdot (hx) = (gh)x \Rightarrow$

$\Rightarrow z \sim x$ ✓

Get $X = \bigsqcup \mathcal{O}$

equivalence classes for $\sim$

to be called **orbits** for the action

Example: $H \curvearrowright G$, $h \cdot g = hg$

$\swarrow$

orbits are **right cosets** Hg

Example: $G \curvearrowright G$, $g \cdot a = gag^{-1}$

orbits are **conjugacy classes**

$S_3 \curvearrowright S_3$

Orbits: $\overset{\overset{S_3}{\curvearrowright}}{\{1\}}$, $\overset{\overset{S_3}{\curvearrowright}}{\{(12),(13),(23)\}}$, $\overset{\overset{S_3}{\curvearrowright}}{\{(123),(132)\}}$

G acts on <u>every</u> orbit $\mathcal{O} \subset X$

↓

"identifies" all elements of this orbit

If $G \curvearrowright X$ with one orbit then this action is called <u>transitive</u>

<u>Example:</u> $G \curvearrowright G/H$

$$g\cdot(aH) = gaH$$

note that H need not be normal!

1) $1 \cdot g_2H = g_2H$ ✓

2) $$\begin{array}{c} g_1\cdot(g_2\cdot aH) = g_1 g_2 a H \\ \| \\ (g_1g_2)\cdot(aH) \end{array} \checkmark$$

Action $G \curvearrowright G/H$ ← **transitive**

It turns out that *any* orbit $\mathcal{O}$ for $G \curvearrowright X$ can be identified with G/H for appropriate H.

Namely, if $x \in X$

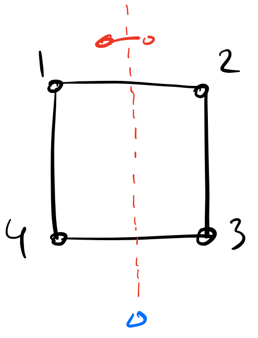
$$\rightsquigarrow$$

stabilizer of x → $G_x := \{ g \in G \mid gx = x \}$

Example $S_4 \curvearrowright \{1,2,3,4\}$

stabilizer of $\{4\}$ is $S_3 \subset S_4$

$D_8 \curvearrowright$



← stabilizer of the segment $[1,2]$ is $\{1, (12)(34)\}$

For $G \curvearrowright G/H$, stabilizer of $H \in G/H$ is $H \subset G$ ↖ so can "read off" H from $G \curvearrowright X = G/H$

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Claim: $\underset{\text{subgroup}}{G_x} \subset G$

Propos. if $\mathcal{O} \subset X$

$\rotatebox{90}{$\in$}$ orbit for $G \curvearrowright X$

and $x \in \mathcal{O}$ $\Rightarrow$ there exists an identification

$$\begin{array}{ccc} G/G_x & \overset{\pi}{\xrightarrow{\sim}} & \mathcal{O} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ gG_x & \longmapsto & g\bullet x \end{array}$$

identification of sets with G-action

proof. 1) $gG_x \overset{\pi}{\longmapsto} g\cdot x$ $\multimap$ is well-defined

take $a \in G_x$, $(ga)\cdot x = g\cdot(a\cdot x) = g\cdot x$ ✓

2) π $\multimap$ surjective: every element of $\mathcal{O}$ is of the form $g\cdot x$ for some $g \in G$ (use that $\mathcal{O}$ $\multimap$ orbit)

3) π $\multimap$ injective: if

$$\begin{array}{ccc} g_1G_x & \longmapsto & g_1\cdot x \\ & & \shortparallel \\ g_2G_x & \longmapsto & g_2\cdot x \end{array}$$

⇙

$$(g_2^{-1} g_1) \circ x = x \Rightarrow g_2^{-1} g_1 \in G_x$$

⇙

$$g_2 G_x = g_2 \cdot (g_2^{-1} g_1) G_x = g_1 G_x \quad ✓$$