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Generators and relations:
general theory

Free group

Start with S an arbitrary set.

"
 $\{x_1, \dots, x_n\}$ may be infinite
in general

$\mathcal{F}(S)$ ← "free" group generated by x_i

↳ consists of all possible products

$x_{i_1}^{\pm 1} \cdot x_{i_2}^{\pm 1} \dots$ modulo relations

$$(*) \quad \dots a \cdot x_i \cdot x_i^{-1} \cdot b \dots = \dots a \cdot b \dots$$

Formal definition of $\mathcal{F}(S)$:

(2)



Let $\tilde{F}(S) \leftarrow \{ x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \dots \}$

define \sim on $\tilde{F}(S)$ as follows:

two elements of $\tilde{F}(S)$ are equivalent if one can be obtained from the other via operations $(*)$.

Claim (see Section 7.9 of Artin's book)
for details

1) \sim is equivalence relation

2) every equivalence class has unique shortest

representative (called reduced word)



check this!

(3)

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Example: stS is reduced $stt^{-1}S$ is not reducedDefinition

$$\mathcal{F}(S) := \tilde{\mathcal{F}}(S) / \sim$$

↓
set of equivalence classes

Claim: $\mathcal{F}(S)$ has a group structure

given by: if $A, B \in \mathcal{F}(S)$, then

to define $A \cdot B$ pick any $a \in A, b \in B$

then $A \cdot B$ is equivalence class of ab

↓

Check that this operation is well-defined

is it clear what you need to check here?

Relations

Start with $\mathcal{F}(S)$, want to impose some relations $R \subset \mathcal{F}(S)$

we want all of these elements to be equal to 1

How to define $\mathcal{F}(S) / \langle R \rangle$?

Ex: $\langle r, s \rangle / \langle r^4, s^2, sr sr^4 \rangle$

what this quotient means?

Note: $\left. \begin{matrix} r^4 = 1 \\ s^2 = 1 \end{matrix} \right\} \Rightarrow r^4 s^2 = 1 \Rightarrow$ the whole subgroup of $\mathcal{F}(S)$ generated by R

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must be in $\langle R \rangle$



Naively: can try $\mathcal{F}(S) /$ subgroup generated by R
this is not a group
in general
may not be normal!

Correct thing to do

$$\langle R \rangle \subset \mathcal{F}(S)$$

minimal normal subgroup that contains R

And then $\mathcal{F}(S) / \langle R \rangle$ makes sense and has group structure

How to think about $\langle R \rangle$:

$\langle R \rangle \leftarrow$ elements of G that can be obtained from R using a finite sequence of the operations of multiplication, inversion, and conjugation.

Ex: $\langle r^4, s^2, sr sr^{-1} \rangle_R$ contains $\frac{r s r^{-1} s}{\parallel}$
 $s^{-1} s r s r^{-1} s$

The most important property of $\mathcal{F}(S)/\langle R \rangle$

Thm $G \leftarrow$ arbitrary group, then

$$\psi: \mathcal{F}(S)/\langle R \rangle \rightarrow G$$

is the same as:

(7)

(12)

1) $y_i \in G \quad \forall x_i \in S'$
↑ collection of elements in G s.t. ↓

2) $\forall x_{i_1} \dots x_{i_k} \in R, \quad y_{i_1} \dots y_{i_k} = 1$

proof.

① given y_i , we can define

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\sim \varphi} & G \\ \downarrow & & \downarrow \\ x_i & \xrightarrow{\quad} & y_i \end{array}$$

it extends to a homomorphism ↪ why?

② We have:

$$F(S) \xrightarrow{\psi} G$$

$$\psi \swarrow$$

$$F(S)/\langle R \rangle$$

$$\nearrow \alpha$$

want ψ s.t.
this diagram
is commutative

General theorem that implies ②

Thm Let $f: S \rightarrow S'$ group homom. with kernel K

Let $N \subset S$ normal contained in K .

Then there is a unique map $S/N \xrightarrow{h} S'$ s.t.

$$S \xrightarrow{f} S'$$

$$\swarrow \searrow$$

$$S/N \xrightarrow{h} S'$$

is commutative

~~will prove next time~~

(9)

proof

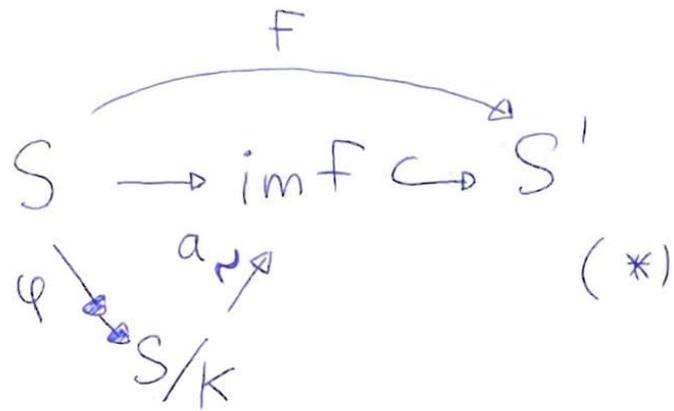
We proved that:

there exists isomorphism

$$\alpha: S/K \xrightarrow{\cong} \text{im } f$$

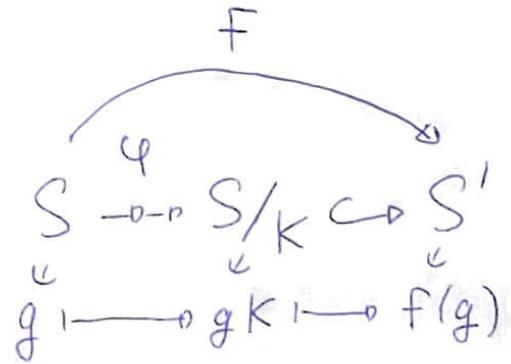
$$\downarrow \qquad \qquad \downarrow$$

$$gK \longmapsto f(g)$$



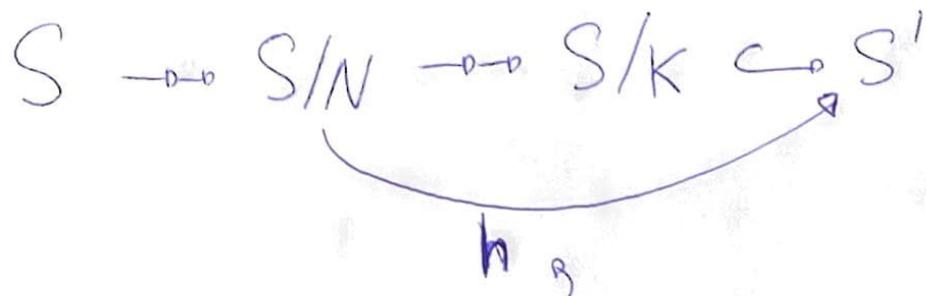
s.t. diagram (*) is commutative.

In other words, F identifies with



Using that $N \subset K$ we conclude that φ factors

through:



this is nothing else
but h

Back to our goal:

$$\begin{array}{ccc}
 \mathcal{F}(S) & \xrightarrow{\tilde{\Psi}} & G \\
 \downarrow \varphi & & \uparrow \cong \\
 & & \mathcal{F}(S)/\langle R \rangle
 \end{array}$$

want Ψ s.t.
this diagram
is commutative

Apply theorem to $F = \tilde{\Psi}$, $N = \langle R \rangle$.

Remains to check that $\ker \tilde{\Psi}$ contains $\langle R \rangle$.

Indeed: 1) $\ker \tilde{\Psi}$ contains R

(remember that any element of R becomes
equal to 1 in G)

2) $\ker \tilde{\Psi}$ is normal

1), 2) $\Rightarrow \langle R \rangle \subset \ker \tilde{\Psi}$ as desired

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Claim $D_{2n} = \langle \Gamma, S \rangle / \langle \Gamma^n, S^2, S\Gamma S\Gamma \rangle$

proof.

By theorem above, ~~map~~ homomorphism

$$\langle \Gamma, S \rangle / \langle \Gamma^n, S^2, S\Gamma S\Gamma \rangle \xrightarrow{\Psi} D_{2n}$$

is well-defined

1) it is surjective $\leftarrow \Gamma, S$ generate D_{2n}

2) remains to check that $|\langle \Gamma, S \rangle / \langle \Gamma^n, S^2, S\Gamma S\Gamma \rangle| \leq 2n$

then Ψ must be an isomorphism \leftarrow clear?

(we proved that $|D_{2n}| = 2n$)

Exercise: every element of $\langle \Gamma, S \rangle / \langle \Gamma^n, S^2, S\Gamma S\Gamma \rangle$ is

of the form Γ^k or $S\Gamma^k \Rightarrow$ have $\leq 2n$ elements in this group