# Generators and relations: general theory

## Free group

Start with $S$ – arbitrary set.

$= \{x_1, \dots, x_n\}$ – may be infinite in general

$\mathcal{F}(S)$ – "free" group generated by $x_i$

consists of all possible products

$$x_{i_1}^{\pm 1} \cdot x_{i_2}^{\pm 1} \cdot \dots \qquad \text{modulo relations}$$

$$(*) \qquad \dots a \cdot x_i \cdot x_i^{-1} \cdot b \dots = \dots a \cdot b \dots$$

Formal definition of $\mathcal{F}(S)$:

Let $\tilde{\mathcal{F}}(S) = \{ x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \dots \}$

define $\sim$ on $\tilde{\mathcal{F}}(S)$ as follows:

two elements of $\tilde{\mathcal{F}}(S)$ are equivalent if one can be obtained from the other via operations $(*)$.

<u>Claim</u> (see Section 7.9 of Artin's book for details)

1) $\sim$ = equivalence relation

2) every equivalence class has <u>unique</u> <u>shortest</u> representative (called <u>reduced</u> word)

check this!

**Example:**

$sts \rightsquigarrow$ reduced

$stt^{-1}s \rightsquigarrow$ not reduced

**Definition** $\mathcal{F}(S) := \tilde{\mathcal{F}}(S)/\sim$

set of equivalence classes

**Claim:** $\mathcal{F}(S)$ has a group structure

given by: if $A, B \in \mathcal{F}(S)$, then

to define $A \cdot B$ pick any $a \in A$, $b \in B$

then $A \cdot B \rightsquigarrow$ equivalence class of $ab$

Check that this operation is well-defined

is it clear what you need to check here?

# Relations

Start with $\mathcal{F}(S)$, want to impose some relations $R \subset \mathcal{F}(S)$

want all of these elements to be equal to 1

How to define $\mathcal{F}(S)/\langle R \rangle$ ?

Ex: $\langle r, s \rangle / \langle r^4, s^2, srsr^{4} \rangle$

what this quotient means?

Note: $\left.\begin{matrix} r^4 = 1 \\ s^2 = 1 \end{matrix}\right\} \Rightarrow r^4 s^2 = 1 \Rightarrow$ the whole subgroup of $\mathcal{F}(S)$ generated by R must be in $\langle R \rangle$

__Naively:__ can try $\mathcal{F}(S)/$ subgroup generated by $R$

this is __not__ a group in general

may not be normal!

Correct thing to do:

$$\langle R \rangle \subset \mathcal{F}(S)$$

minimal __normal__ subgroup that contains $R$

And then $\mathcal{F}(S)/\langle R \rangle$ ← makes sense and has group structure

How to think about $\langle R \rangle$:

$\langle R \rangle$ – elements of $G$ that can be obtained from $R$ using a finite sequence of the operations of multiplication, inversion, and <u>conjugation</u>.

<u>Ex:</u> $\langle r^4, s^2, srsr \rangle$ R contains $rsrs$

$$\underline{rsrs} = s^{-1} \cdot srsr \cdot s$$

<u>The most important property of $\mathcal{F}(S)/\langle R \rangle$</u>

<u>Thm</u> $G$ – arbitrary group, then

$\varphi : \mathcal{F}(S)/\langle R \rangle \to G$

is <u>the same</u> as:

1) $y_i \in G \quad \forall x_i \in S$

collection of elements in $G$ s.t.

2) $\forall\, x_{i_1} \cdots x_{i_k} \in R$, $y_{i_1} \cdots y_{i_k} = 1$

<u>proof.</u>

1 given $y_i$, we can define

$$\begin{array}{ccc} \mathcal{F}(S) & \xrightarrow{\tilde{\psi}} & G \\ \rotatebox[origin=c]{90}{$\in$} & & \rotatebox[origin=c]{90}{$\in$} \\ x_i & \longmapsto & y_i \end{array}$$

it extends to a homomorphism why?

2 We have:

$$\begin{tikzcd} \mathcal{F}(S) \arrow[rr, "\tilde{\Psi}"] \arrow[dr, "\varphi"'] & & G \\ & \mathcal{F}(S)/\langle R\rangle \arrow[ur] & \end{tikzcd}$$

want $\Psi$ s.t. this diagram is commutative

General theorem that implies 2

<u>Thm</u> Let $f: S \to S'$ group homom. with kernel $K$

Let $N \subset S$ normal contained in $K$.

Then there is a <u>unique</u> map $S/N \xrightarrow{h} S'$ s.t.

$gN \mapsto f(g)$

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr] & & S' \\ & S/N \arrow[ur, "h"'] & \end{tikzcd}$$

is commutative

~~will prove next time~~

proof

We proved that:

there exists isomorphism

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\varphi"'] \arrow[rr, bend left, "F"] & \operatorname{im} f \arrow[r, hook] & S' \\ S/K \arrow[ur, "a", "\sim"'] & & \end{tikzcd} \quad (*)$$

$$\begin{aligned} a : S/K &\xrightarrow{\sim} \operatorname{im} f \\ \rotatebox{90}{$\in$} \quad & \quad \rotatebox{90}{$\in$} \\ gK &\longmapsto f(g) \end{aligned}$$

s.t. diagram $(*)$ is commutative.

In other words, $f$ identifies with

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[rr, bend left, "F"] & S/K \arrow[r, hook] & S' \\ g \arrow[r, mapsto] & gK \arrow[r, mapsto] & f(g) \end{tikzcd}$$

Using that $N \subset K$ we conclude that $\varphi$ factors through:

$$\begin{tikzcd} S \arrow[r] & S/N \arrow[r] \arrow[rr, bend right, "h"'] & S/K \arrow[r, hook] & S' \end{tikzcd}$$

this is nothing else but $h$

Back to our goal:

$$\begin{tikzcd} \mathcal{F}(S) \arrow[r, "\tilde{\varphi}"] \arrow[dr, "\varphi"'] & G \\ & \mathcal{F}(S)/\langle R\rangle \arrow[u] \end{tikzcd}$$

want $\Psi$ s.t. this diagram is commutative

Apply theorem to $f = \tilde{\varphi}$, $N = \langle R \rangle$.

Remains to check that $\ker \tilde{\varphi}$ contains $\langle R \rangle$.

Indeed: 1) $\ker \tilde{\varphi}$ contains $R$

(remember that any element of $R$ becomes equal to $1$ in $G$)

2) $\ker \tilde{\Psi}$ normal

1), 2) $\Rightarrow$ $\langle R \rangle \subset \ker \tilde{\varphi}$ as desired

Claim $D_{2n} = \langle r, s \rangle / \langle r^n, s^2, srsr \rangle$

proof

By theorem above, ~~map~~ homomorphism

$$\langle r,s \rangle / \langle r^n, s^2, srsr \rangle \xrightarrow{\psi} D_{2n}$$

is well-defined

1) it is surjective $\leftarrow$ $r, s$ generate $D_{2n}$

2) remains to check that $|\langle r,s \rangle / \langle r^n, s^2, srsr \rangle| \leq 2n$

then $\psi$ must be an isomorphism $\leftarrow$ clear?

(we proved that $|D_{2n}| = 2n$)

Exercise: every element of $\langle r,s \rangle / \langle r^n, s^2, srsr \rangle$ is of the form $r^k$ or $sr^k$ $\Rightarrow$ have $\leq 2n$ elements in this group