Lecture 9

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Last time I mentioned the following fact:

If G ⟜ semisimple simply-connected group acting on

enough to assume normal ↘ only need for uniqueness ↙

a smooth, projective variety X, and $\mathcal{L}$ is a line bundle on X, then $\mathcal{L}$ has a **unique** G-equivariant structure.

prf. (idea)

1) Assume that $\mathcal{L}$ has two G-equiv. structures $\mathcal{L}_1$, $\mathcal{L}_2$, then we need to check that the identity isomorphism $\mathcal{L}_1 \xrightarrow{\sim} \mathcal{L}_2$ is G-equivariant.

$$\underset{\underset{G}{\circlearrowright}}{\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2)} = \Gamma(X, \mathcal{L}_2 \otimes \mathcal{L}_1^{-1}) = \mathbb{C}$$

$\Downarrow$

G is a one-dimensional representation of G, it must be trivial (use that G-semisimple)

see Chriss-Ginzburg "Representation theory and Complex Geometry" Thm 5.1.9

2) Consider the action map $a: G \times X \to X$

and the projections $p_G : G \times X \to G$

$p_X : G \times X \to X$

Set $E := a^* \mathcal{L}|_{\{1\} \times X} \simeq \mathcal{L}$

Using tha G contains an open subset isomorphic to $(\mathbb{C}^\times)^r \times (\mathbb{A}^1)^k$ $\left(= U_- \times T \times U \right)$ one can show that:

$$a^* L \simeq (p_G^* F) \otimes (p_X^* E)$$

for some line bundle F on G.

Group G is simply-connected $\Rightarrow \mathrm{Pic}(G) = 0$

so $F \simeq \mathcal{O}_G$ is the trivial bundle.

We conclude that $a^* \mathcal{L} \simeq p_X^* \mathcal{L}$.

Then one can show that this identification can be modified in such a way that it gives desired G-equivariant structure on $\mathcal{L}$.

Last time:

$\{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \Delta$ simple roots $\rightsquigarrow \mathfrak{m} \subset \mathfrak{g}$

Levi subalgebra

$S_\mathfrak{m} := S_e \quad , e \in \mathfrak{m}$ regular

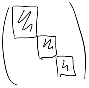
$\tilde{S}_\mathfrak{m} = \tilde{S}_e$

$$\begin{tikzcd} Y = \tilde{S}_\mathfrak{m} \arrow[d] & Y^\vee = T^*(G^\vee/P^\vee) \arrow[d] \\ X = S_\mathfrak{m} & X^\vee = \operatorname{Spec} \mathbb{C}[T^*G^\vee/P^\vee] \end{tikzcd}$$

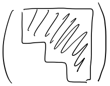
$\mathfrak{m} \rightsquigarrow \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \Delta \simeq \Delta^\vee \supset \{\alpha^\vee_{i_1}, \dots, \alpha^\vee_{i_k}\} \rightsquigarrow \mathfrak{m}^\vee \subset \mathfrak{g}^\vee$

$M^\vee \subset P^\vee = M^\vee \cdot B^\vee$

parabolic for $M^\vee$



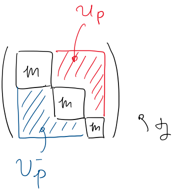
$\Rightarrow P^\vee =$



Let's understand better $Y^\vee, X^\vee$ as above.

Will not write checks anymore.

$$T^*(G/P) = G \times^P \mathfrak{p}^\perp = G \times^P \mathfrak{u}_\mathfrak{p}$$



invariant pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$

pairs $\mathfrak{m} \times \mathfrak{m} \to \mathbb{C}$ ← nondegenerate

$\mathfrak{u}_\mathfrak{p}^- \times \mathfrak{u}_\mathfrak{p} \to \mathbb{C}$

$$\Downarrow$$

$\mathfrak{p}^\perp \simeq \mathfrak{u}_\mathfrak{p}$ under $(\ ,\ )$

So, we have a map:

$$\begin{tikzcd} T^*(G/P) = G \times^P \mathfrak{u}_\mathfrak{p} \arrow[d, "\pi"'] \\ \mathcal{N} \end{tikzcd}$$

This map is G-invariant, image of π is irreducible $\Rightarrow$ $\operatorname{im} \pi = \overline{\mathbb{O}_\mathfrak{p}}$

use that number of G-orbits on $\mathcal{N}$ is finite

some nilpotent orbit

Prop. if $\mathfrak{g} = \mathfrak{sl}_n$, then morphism π is a resolution of singularities and

$$\overline{\mathbb{O}_p} = \operatorname{Spec} \mathbb{C}[T^*(G/P)] = X$$

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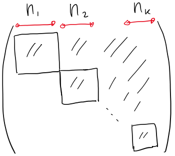
prf.

1) π ← proper (Same proof as for the Springer resolution)

2) π ← iso over $\mathbb{O}_p \subset \overline{\mathbb{O}_p}$

$P =$



, $n_1 \geq n_2 \geq \dots$

$\Downarrow$

Q

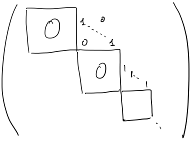
$$T^*(G/P) = \left\{ \begin{array}{l} \{0\} = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_{k-1} \subset F_k = \mathbb{C}^n \\ x \in \mathfrak{g} \end{array} \middle| \begin{array}{l} \dim F_i - \dim F_{i-1} = n_i \\ x(F_i) \subset F_{i-1} \end{array} \right\}$$

$$\begin{tikzcd} T^*(G/P) \arrow[d] & \\ \mathcal{N} \ni x & \end{tikzcd}$$

$x \mapsto x$

Orbit $\mathbb{O}$ is the orbit of the element:

$e =$



$\in \mathcal{U}_p$

Let's describe the fiber $\pi^{-1}(e)$:

Assume $F_\bullet \in \pi^{-1}(e)$

$$\Downarrow$$

$$e^i(F_i) = 0 \Rightarrow F_i \subset \ker e^i$$

$$\dim F_i = n_1 + \dots + n_i = \dim e^i$$

easy to see from the explicit form of e

$$\Downarrow$$

$F_i = \ker e^i$ and π is an iso. over $\mathbb{O}_e$

So $\pi : T^*(G/P) \to \overline{\mathbb{O}_e}$ ← resolution of sing

3) $\overline{\mathbb{O}_e}$ ← normal (nontrivial)

$$\Downarrow$$

$$\mathbb{C}[\overline{\mathbb{O}}_e] = \mathbb{C}[T^*(G/P)] \Rightarrow$$

$$\Rightarrow \overline{\mathbb{O}_e} = X$$

Rmk: $\overline{\mathbb{O}_e} = \operatorname{Spec} \mathbb{C}[\mathbb{O}_e]$

use normality of $\overline{\mathbb{O}}_e$ + $\underset{\partial\overline{\mathbb{O}}_e}{\operatorname{codim}} \overline{\mathbb{O}_e} \geq 2$

(nilp. orbits are even-dim)

It is NOT true that for arbitrary

$\mathfrak{g}$, $\overline{\mathbb{O}}_e$ ← normal.

On the other hand, $\operatorname{Spec}\mathbb{C}[\mathbb{O}]$ ↗ $\underline{\underline{\text{is}}}$ normal

(because $\mathbb{O}$ -smooth ⇒ ⇒ normal)

We have

$$\begin{tikzcd} & \operatorname{Spec}\mathbb{C}[\mathbb{O}] \arrow[d] \\ \mathbb{O} \arrow[ur] \arrow[r, hook] & \overline{\mathbb{O}} \end{tikzcd}$$

so this is the "correct replacement" of $\overline{\mathbb{O}}$ in general

normalization of $\overline{\mathbb{O}}$

induced by the map $\mathbb{C}[\overline{\mathbb{O}}] \to \mathbb{C}[\mathbb{O}]$

iso over $\mathbb{O}$

G-equivariant

Now let's describe $T^*(G/P) \to \operatorname{Spec}\mathbb{C}[T^*G/P]$

for arbitrary G.

Consider $P \curvearrowright U_p$, let $\mathbb{O}_p \subset \mathcal{N}$ ← nilpotent orbit

be s.t. $\mathbb{O}_p \cap U_p$ ← dense, open.

Q

such $\mathbb{O}_p$ exists and unique (use that number of G-orbits on $\mathcal{N}$

is finite)

We will use two general facts about parabolic groups (see Carter's book "Finite groups of Lie type: conjugacy classes and complex characters" 5.2.3/4)

(A) $\mathbb{O}_P \cap \mathcal{U}_P$ is the unique P-orbit on $\mathcal{U}_P$

(B) if $e \in \mathbb{O}_P \cap \mathcal{U}_P$
$\Downarrow$
$$Z_G(e)^\circ \subset Z_P(e) \subset Z_G(e)$$

Set $\mathcal{U}_P^{reg} := \mathbb{O}_P \cap \mathcal{U}_P$

$$\begin{tikzcd}
T^*(G/P) \arrow[r, phantom, "="] & G \times^P \mathcal{U}_P \arrow[dd] \arrow[r, phantom, "\supset"] & G \times^P \mathcal{U}_P^{reg} \arrow[dd, "p"] \arrow[r, phantom, "="] & G \times^P (P/Z_P(e)) \arrow[d, equal] \\
 & & & G/Z_P(e) \arrow[d] \\
 & \overline{\mathbb{O}} \arrow[r, phantom, "\supset"] & \mathbb{O} \arrow[r, phantom, "="] & G/Z_G(e)
\end{tikzcd}$$

fibers are $Z_G(e)/Z_P(e)$ ← finite

So,

$$\begin{tikzcd} G \times^P \mathcal{U}_p^{reg} \arrow[r] \arrow[d, equal] & \mathbb{O} \\ \widetilde{\mathbb{O}} & \end{tikzcd}$$

G-equivariant finite cover

Comment on the proof of (B): one shows that

$\dim \mathbb{O} = \dim T^*(G/P) \Rightarrow G \times^P \mathcal{U}_p^{res} \to \mathbb{O}$ must be G-equivariant finite morphism $\Rightarrow \operatorname{Lie} Z_P(e) = \operatorname{Lie} Z_G(e)$

$$\Downarrow$$

$$Z_P(e)^\circ = Z_G(e)^\circ$$

$$\Downarrow$$

$$(B)$$

Prop: $T^*(G/P) \to \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}] = X$ is a conical symplectic resolution of singularities

Will not prove this carefully, ideas:

$$\begin{tikzcd} T^*(G/P) \arrow[d] & \supset & \widetilde{\mathbb{O}}_p \arrow[d] \\ \operatorname{Spec} \mathbb{C}[T^*(G/P)] \arrow[d] & & \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}_p] \arrow[ll] \\ \overline{\mathbb{O}}_p & & \end{tikzcd}$$

(as $\widetilde{\mathbb{O}}_p \to \mathbb{O}_p$ - finite) finite onto the image

$$\Downarrow$$

open embedding

iso as $\operatorname{codim} T^*(G/P) \setminus \widetilde{\mathbb{O}}_p \geq 2$

Conical action: $\tilde{\mathbb{O}}_p \simeq G/H$.

In general, $\mathrm{Aut}_G(G/H) \overset{\text{Ex}}{=} \underset{\substack{\rotatebox{90}{$\in$} \\ n}}{N_G(H)}/H$.

$$n \cdot gH = gn^{-1}H$$

Pick $e \in \mathbb{O}_p$, extend to $\mathfrak{sl}_2$-triple e, h, f,

obtain $t^h : \mathbb{C}^\times \to G$.

Ex: t^h normalizes $Z_G(e)$ and any finite index sbgrp.

Pick $\tilde{e} \in \tilde{\mathbb{O}}_p$ over $e \Rightarrow \tilde{\mathbb{O}}_p = G/\underset{\substack{\uparrow \\ Z_G(\tilde{e})}}{H}$ and

we get the desired $\mathbb{C}^\times$-action.

The fact that the action is conical $\overset{\text{Ex}}{\text{follows}}$

from the fact that

$$\begin{tikzcd} \mathbb{C}[\overline{\mathbb{O}}_p] \arrow[rr, bend right, "\text{finite extension}"'] & \subset \mathbb{C}[\mathbb{O}_p] \subset & \mathbb{C}[\tilde{\mathbb{O}}_p] \end{tikzcd}$$

First example when cover $\tilde{\mathbb{O}}$ indeed

appears: $\mathfrak{g} = \mathfrak{sp}_4$, the skew-symmetric

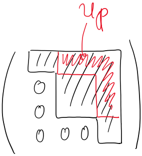
form on $\mathbb{C}^4 = \langle v_1, v_2, v_3, v_4 \rangle$ is: $(v_1, v_4) = 1$, $(v_4, v_1) = -1$

$(v_2, v_3) = 1$, $(v_3, v_2) = -1$

$$Sp_4 = \left(\begin{array}{cc|cc} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{13} \\ \hline a_{31} & a_{32} & -a_{22} & -a_{12} \\ a_{41} & a_{31} & -a_{21} & -a_{11} \end{array}\right) \qquad Q = \left(\begin{array}{cc|cc} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right)$$

$$\mathfrak{m} = \mathfrak{gl}_1 \oplus \mathfrak{sp}_2 = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \boxed{\begin{matrix} b & c \\ d & -b \end{matrix}} & & 0 \\ 0 & & & 0 \\ 0 & 0 & 0 & -a \end{pmatrix}$$

$\mathfrak{p} =$



$\subset \mathfrak{sp}_4$

$$e = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$Z_G(e) \supset Z_P(e)$, $Z_G(e)$ contains the element

proper!

$v_1 \mapsto v_3$

$v_4 \mapsto -v_2$

$v_2 \mapsto -v_4$

$v_3 \mapsto v_1$

$$\rightarrow \quad s = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \notin P$$

One can show that in this case:

$$T^*(G/P) \longrightarrow \mathbb{O}$$

2 : 1 over $\mathbb{O}$

Why same nilpotent is "fine" considered as an element of $\mathfrak{sl}_4$: in basis

$$e = \left(\begin{array}{cccc} V_3 & V_1 & V_2 & V_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

this is our new parabolic,

it contains $s = \left(\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right)$

Upshot:

$$\begin{tikzcd} Y = \tilde{S}_m \arrow[d] & \longleftrightarrow & T^*(G^\vee/P^\vee) = Y^\vee \arrow[d] \\ S_m & & \operatorname{Spec} \mathbb{C}[\tilde{\mathbb{O}}_{p^\vee}] \end{tikzcd}$$

equal to $\overline{\mathbb{O}}_{p^\vee}$ in type A

How to make duality look more symmetric.

Let's introduce a family of varieties depending on a pair $\mathfrak{p}, \mathfrak{a} \subset$ parabolic in $\mathfrak{g}$, generalizing both Y and $Y^\vee$.

Parabolic Slodowy variety:

$$\begin{tikzcd}
\tilde{S}^{\mathfrak{a}}_{\mathfrak{p}} \arrow[r, hook] \arrow[d] & T^*(G/A) \arrow[d, "\pi"] \\
S_{\mathfrak{p}} \arrow[r, phantom, "\subset"] & \mathcal{N} \\
\tilde{S}^{\mathfrak{a}}_{\mathfrak{p}} := \pi^{-1}(S_{\mathfrak{p}}) = Y \arrow[uu, bend left] &
\end{tikzcd}$$

$$\begin{array}{c} \operatorname{Spec} \mathbb{C}[\tilde{S}^{\mathfrak{a}}_{\mathfrak{p}}] =: X \\ \| \\ S^{\mathfrak{a}}_{\mathfrak{p}} \end{array}$$

people didn't study this family much outside of type A!

Prop: (a) $\tilde{S}_p^a \to S_p^a$ ← symplectic resolution of sing

(b) If $\mathfrak{g} = \mathfrak{sl}_n \Rightarrow S_p^a = S_p \cap \overline{\mathbb{O}}_a$

prf. (only very rough idea)

recall: $S_p = \mathcal{N} /\!/\!/_\chi Q$, $\tilde{S}_p = T^*(G/B) /\!/\!/_\chi Q$

Same arguments show that:

$$\tilde{S}_p^a = \underset{\substack{\uparrow \\ \text{smooth} \\ \text{symplectic}}}{T^*(G/A)} /\!/\!/_\chi Q \quad ; \quad S_p^a = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}_a] /\!/\!/_\chi Q$$

Now the fact that $\tilde{S}_p^a \to S_p^a$ is a resolution follows from the fact that

$T^*(G/A) \to \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}_a]$ is a resolution

Rem: previously, we had two arguments proving that $\widetilde{S}_m$ is smooth (with Hamiltonian reduction by Q and without), now we have only one.

Again: $\widetilde{S}_p^a$ ← very interesting symplectic varieties that people studied mostly in type A

From the definitions:

$$\widetilde{S}_p^b \overset{Q}{=} \widetilde{S}_p \;, \quad \widetilde{S}_b^a = T^*(G/A)$$

3d mirror symmetry for parabolic Slodowy becomes very natural:

$$Y = \widetilde{S}_p^a \xleftrightarrow{\text{dual}} Y^\vee \overset{Q}{=} \widetilde{S}_{a^\vee}^{p^\vee}$$