Lecture 8

четверг, 27 февраля 2025 г. 12:58

Last time:

← Before!

Started to discuss 3d mirror symmetry

$$\begin{tikzcd} X \arrow[rr, leftrightarrow, "\text{3d dual}"] & & X^\vee \\ T_X \arrow[u, phantom, "\circlearrowright"] & & T_{X^\vee} \arrow[u, phantom, "\circlearrowright"] \end{tikzcd}$$

conical symplectic sing.

$\mathfrak{h}_X := H^2(Y, \mathbb{C})$, $\mathfrak{h}_{X^\vee} := H^2(Y^\vee, \mathbb{C})$

one can define this space even if there is no symplectic resolution Y

predictions:

$\mathfrak{h}_X \simeq \mathfrak{t}_{X^\vee}$

$\mathfrak{h}_{X^\vee} \simeq \mathfrak{t}_X$

$X^\vee$ - has symplectic resolution $\Leftrightarrow$ $X^{T_X} = \{p_X\}$ $(*)$

$Y^{T_X} \simeq (Y^\vee)^{T_{X^\vee}}$ ← equality of finite sets

Ex (from last time)

$Y = \tilde{S}_e \to S_e = X \leftrightarrow T^*\mathbb{P}^{n-1} \to \overline{\mathbb{O}}_{min}$

e subregular in $\mathfrak{sl}_n$

More generally, let's consider:

$$Y = \tilde{S}_e \to S_e = X$$

Question: who is $X^\vee$?

Before answering this question let's describe S_e for which the dual should have a symplectic resolution.

Question: when $S_e^{T_X} = \{e\}$?

$$\mathrm{Aut}^{\mathbb{C}^\times}_{\{\,,\,\}}(X)^\circ = Z_G(e,h,f)^\circ$$

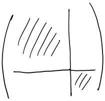
$$\Downarrow$$

$T_X = T_e$ ← max. torus in $Z_G(e,h,f)$

Set: $M := Z_G(T_e)$ ← Levi subgroup of G

$\mathfrak{m} := \mathrm{Lie}\, M$

Ex: e ← subregular, $G = PGL_n \Rightarrow T_e = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & \\ & & 1 & \\ \hline & & & t \end{array}\right)$

$M \overset{Q}{=}$ 

Note: $e, h, f \in \mathfrak{m}$

Claim: $S_e^{T_e} = \{e\} \Leftrightarrow e \in \mathfrak{m}$ ← regular (considered as element of $\mathfrak{m}$)

prf. $M = \{ g \in G \mid tgt^{-1} = g \quad \forall t \in T_e \}$

$\Downarrow$

$\mathfrak{m} = \{ x \in \mathfrak{g} \mid Ad_t(x) = x \quad \forall t \in T_e \}$

$\|$

$\mathfrak{g}^{T_e}$

So $\mathcal{N}(\mathfrak{m}) = \mathcal{N}(\mathfrak{g})^{T_e}$

nilpotent cone for $\mathfrak{m}$

We see that:

$$S_e^{T_e} = S_e \cap \mathcal{N}(\mathfrak{m}) = (e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{N}(\mathfrak{m}) = S_e(\mathfrak{m})$$

$\dim S_e(\mathfrak{m}) = \dim \mathcal{N}(\mathfrak{m}) - \dim \mathbb{O}_e(\mathfrak{m}) \curvearrowright$ is zero iff e is regular in $\mathfrak{m}$

Prop. assume $\mathfrak{g} = \mathfrak{sl}_n$ then any e is regular in the corresponding Levi $\mathfrak{m}$.

prf. let's first of all describe the group $Z_{GL_n}(e,h,f)$ in this case

$\langle e,h,f \rangle \hookrightarrow \mathfrak{sl}_n \curvearrowright \mathbb{C}^n$

is

$\mathfrak{sl}_2$

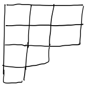
decompose $\mathbb{C}^n = \bigoplus V_i^{a_i}$ $\quad (V_i = S^i\mathbb{C}^2)$

e has a_i Jordan blocks of size $i+1$

$$Z_{GL_n}(e,h,f) = \mathrm{Aut}_{\mathfrak{sl}_2}(\mathbb{C}^n) = \prod GL_{a_i}$$

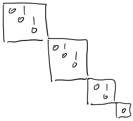
where a_i is the number of occurence of $i+1$ in the partition that corresponds to e:

Ex:



$= e \implies Z_{GL_9}(e,h,f) = GL_2 \times GL_1 \times GL_1$

‖



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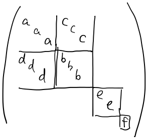
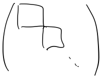
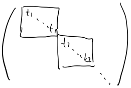


We see that $T_e = \prod T_{a_i}$ ← max torus in GL_{a_i}

Explicitly: $e =$



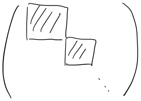
$\implies T_e =$



see example above

⇓

$\mathfrak{m} =$



$\ni e$ ← is indeed regular

Claim: in general, e will not be regular in $\mathfrak{m}$

Ex. $\mathfrak{g} = DE$, e – subregular

Then $X = S_e = \mathbb{A}^2/\Gamma$

Exer.

We claim that $G_X = \mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(X)$ is finite

This would imply $T_X = \{1\} \Rightarrow M = G$ and

e is NOT regular in $\mathfrak{m}$

time permit.

Sketch of the proof (that G_X - finite)

① $X^{reg} := (\mathbb{A}^2 \setminus \{0\})/\Gamma$ – smooth symplectic

Q use that X - normal

② $G_X \xrightarrow{\sim} G_{X^{reg}}$, enough to check

that $\mathrm{Lie}\, G_{X^{reg}} = \{0\}$

③ $\mathbb{A}^2 \setminus \{0\} \twoheadrightarrow X^{reg}$

symplectic, étale cover

hint: pick $s \in \Gamma$, $s \neq \pm 1$

then $Z_{SL_2}(s) = \mathbb{C}^\times$

$$\mathrm{Lie}\, G_{X^{reg}} \overset{''}{\hookrightarrow} \left(\mathrm{Lie}\, G_{\mathbb{A}^2\setminus\{0\}}\right)^\Gamma = \mathfrak{sl}_2^{\Gamma} = \mathrm{Lie}\, Z_{SL_2}(\Gamma) = 0$$

Upshot: for $X = S_e$ the dual $X^\vee$

should have a symplectic resolution iff

$e \in \mathfrak{m}$ is regular.

This is always the case in type A but may fail in other types.

Important remark:

$$\{e \text{ s.t. } e \in \mathfrak{m}\text{-regular}\}/G \longleftrightarrow \{\text{Levi subalgebras of } \mathfrak{g}\}/G$$

Bijection is given by:

$$e \longmapsto \mathfrak{m}$$

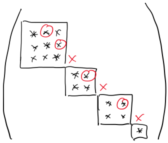
$$\text{regular nilp. element in } \mathfrak{m} \longmapsfrom \mathfrak{m}$$

Every Levi $\mathfrak{m} \subset \mathfrak{g}$ is conjugated to the standard one $\mathfrak{m}'$, standard Levi are labeled by

$\{\alpha_{i_1}, \dots, \alpha_{i_k}\} \longleftarrow$ subsets of the set of simple roots for $\mathfrak{g}$

$$\mathfrak{m}' = \mathfrak{t} \oplus \bigoplus_{\alpha \in \underset{\text{roots of } \mathfrak{g}}{\Delta} \cap \operatorname{Span}_{\mathbb{C}}\langle \alpha_{i_1}, \dots, \alpha_{i_k}\rangle} \mathfrak{g}_\alpha$$

Ex:



$$\longleftrightarrow \{\alpha_1, \alpha_2, \alpha_4, \alpha_6\} \subset \Delta$$

For arbitrary standard Levi $\mathfrak{m} \subset \mathfrak{g}$, set

$S_{\mathfrak{m}} := S_e$, where $e \in \mathfrak{m}$ a regular

Goal: construct the dual to

$$Y = \tilde{S}_{\mathfrak{m}} \to S_{\mathfrak{m}} = X$$

Recall that there should be a resolution

$Y^\vee$ so we will construct it and then

$X^\vee = \operatorname{Spec} \mathbb{C}[Y^\vee]$.

Let's guess the answer.

Start with $\mathfrak{m} = 0 \Rightarrow \tilde{S}_e \overset{Q}{=} T^*(G/B)$.

time!

Whant $X^\vee$ s.t.

$$\mathfrak{t}_{X^\vee} = H^2(T^*(G/B)) = H^2(G/B) \overset{???}{=} \mathfrak{t}_{\mathfrak{g}}^*$$

follows from the fact that $H^i(G/B, \mathcal{O}) = 0$ (particular case of Borel-Weyl-Bott thm)

1) $H^2(G/B) \xleftarrow[\sim]{c_1} \operatorname{Pic}(G/B) \otimes_{\mathbb{Z}} \mathbb{C}$

2) Can assume G a simply-connected (if not, replace G by its simply-connected cover $\tilde{G}$ and B by its preimage in $\tilde{G}$)

3) Then $\mathrm{Pic}_G(G/B) \simeq \mathrm{Pic}(G/B)$

↳ G-equivariant line bundles

4) $\mathrm{Pic}_G(G/B) = \mathrm{Pic}_B(pt) = \mathrm{Hom}(B, \mathbb{C}^\times) = \mathrm{Hom}(T, \mathbb{C}^\times)$

5) $\mathrm{Hom}(T, \mathbb{C}^\times) \otimes_{\mathbb{Z}} \mathbb{C} = \mathrm{Hom}(\mathfrak{t}, \mathbb{C}) = \mathfrak{t}^*_{\mathfrak{g}}$

$$H^2(X^\vee) = \mathfrak{t}_X = \mathfrak{t}_{\mathfrak{g}}$$

$$X^{T_G} = (G/B)^{T_G} = W \leftrightarrow (X^\vee)^{T_{G^\vee}}$$

It's natural to guess that:

$$X^\vee \overset{Q}{=} T^*(G^\vee/B^\vee)$$

In general:

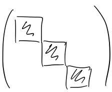
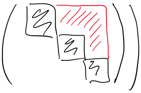
$$Y = \tilde{S}_{\mathfrak{m}} \leftrightarrow Y^\vee = T^*(G^\vee/P^\vee) \to \mathrm{Spec}\, \mathbb{C}[T^*(G^\vee/P^\vee)] =: X^\vee$$

$$\mathfrak{m} \leftrightarrow \{\alpha_{i_1}, \dots, \alpha_{i_k}\} \subset \Delta \simeq \Delta^\vee \supset \{\alpha^\vee_{i_1}, \dots, \alpha^\vee_{i_k}\} \leadsto \mathfrak{m}^\vee \subset \mathfrak{g}^\vee$$

$$M^\vee \subset P^\vee$$

↳ parabolic for $M^\vee$, $\mathfrak{p}^\vee = \mathfrak{m}^\vee + \mathfrak{b}^\vee$

will discuss more next time

$(M^\vee =$  $\Rightarrow P^\vee =$  $)$

Next time:

① $\mathfrak{g} = \mathfrak{sl}_n \Rightarrow X^\vee = \overline{\mathbb{O}}_{e^\vee}$ ← closure of
some nilp. orbit

② $\mathfrak{g}$ ← arbitrary $\Rightarrow X^\vee = \operatorname{Spec} \mathbb{C}[\widetilde{\mathbb{O}}_{e^\vee}]$

some G-equiv.
cover of nilp. orbit!
(will consider examples)

③ generalization and more symmetric
formulation of this duality (parabolic Slodowy var.)

④ * Our duality vs BVLS duality

(Barbash-Vogan-Lusztig-Spaltenstein)