Lecture 7

воскресенье, 23 февраля 2025 г. 07:58

Last time:

$e \leftarrow$ subregular in $\mathfrak{sl}_n$, $e = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & 1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}$

Proved: $$S_e \underset{\substack{\uparrow \\ \text{iso. of Poisson var.}}}{\simeq} \mathbb{A}^2/\mathbb{Z}/n\mathbb{Z}$$

What's the profit?

We get an explicit construction of the resolution of $\mathbb{A}^2/\mathbb{Z}/n\mathbb{Z}$:

$$\begin{tikzcd} & \text{symplectic resolution} \arrow[dl] \\ \tilde{S}_e \arrow[d, "\pi"'] & \left\{ (x, F_\bullet) \,\middle|\, \begin{array}{l} x \in S_e \\ x(F_i) \subset F_{i-1} \end{array} \right\} \arrow[l] \\ S_e & \end{tikzcd}$$

$\pi \leftarrow$ iso. over $S_e \setminus \{e\} = (\mathbb{A}^2 \setminus \{0\})/\mathbb{Z}/n\mathbb{Z}$ so to understand $\tilde{S}_e$ we need to describe the fiber $\pi^{-1}(e) =: \mathcal{B}_e \leftarrow$ Springer fiber

Let's do $n=4$ for concreteness

$$\mathcal{B}_e = \{ 0 \subset F_1 \subset F_2 \subset F_3 \subset \mathbb{C}^4 \mid e(F_i) \subset F_{i+1} \}$$

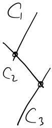
← BEFORE!

$$\mathbb{C}^4 = \langle v_1, v_2, v_3, v_4 \rangle$$

$$e : \begin{array}{l} v_3 \mapsto v_2 \mapsto v_1 \mapsto 0 \\ v_4 \mapsto 0 \end{array}$$

Exercise:

$\mathcal{B}_e =$



$\Big\}\, n-1$, $C_i \simeq \mathbb{P}^1$

$$C_1 = \{ \langle p_1 v_1 + q_1 v_4 \rangle \subset \langle v_1, v_4 \rangle \subset \langle v_1, v_2, v_4 \rangle \subset \mathbb{C}^4 \mid [p_1 : q_1] \in \mathbb{P}^1 \}$$

$$C_2 = \{ \langle v_1 \rangle \subset \langle v_1, p_2 v_2 + q_2 v_4 \rangle \subset \langle v_1, v_2, v_4 \rangle \subset \mathbb{C}^4 \mid [p_2 : q_2] \in \mathbb{P}^1 \}$$

$$C_3 = \{ \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \langle v_1, v_2, p_3 v_3 + q_3 v_4 \rangle \subset \mathbb{C}^4 \mid [p_3 : q_3] \in \mathbb{P}^1 \}$$

3d mirror symmetry

NO definition!

$$\begin{tikzcd} X \arrow[rr, leftrightarrow, "\text{3d dual}"] & & X^\vee \end{tikzcd}$$

symplectic singularities

The best possible situation is when symplectic resolution exists on both sides:

$$Y \to X \quad \leftrightarrow \quad Y^\vee \to X^\vee$$

Ex: $Y = \tilde{S}_e \to S_e = X \quad \leftrightarrow \quad Y^\vee = T^*\mathbb{P}^{n-1} \xrightarrow{\pi^\vee} \{A \in \mathfrak{sl}_n \mid \operatorname{rk} A \leq 1\} = X^\vee$

subregular for $\mathfrak{sl}_n$

will discuss later how to see that this duality is something natural

Rem: $\dim Y = 2$, $\dim Y^\vee = 2(n-1)$, so dual varieties have different dimensions! Unlike varieties appearing in homological mirror symmetry.

How to think about $\pi^\vee$:

1) $X^\vee = \overline{\mathbb{O}_{\min}}$, $\quad \mathbb{O}_{\min} = SL_n \cdot \left(\begin{array}{c|c} \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} & 0 \\ \hline 0 & 0 \end{array}\right)$

$\overline{\mathbb{O}_{\min}} = \{A \in \mathfrak{sl}_n \mid A^2 = 0\}$

2) $\mathbb{P}^{n-1} = \{\{0\} = F_0 \subset F_1 \subset F_n = \mathbb{C}^n \mid \dim F_i = i\}$

Same argument as for the flag variety

shows that:

$$\begin{tikzcd} T^*\mathbb{P}^{n-1} = \arrow[d, "\pi^\vee"'] & \left\{ (X, F_\cdot) \in \mathcal{N} \times \mathbb{P}^{n-1} \;\middle|\; \begin{array}{l} X(\mathbb{C}^n) \subset F_1 \\ X(F_1) = \{0\} \end{array} \right\} \arrow[d, "\pi^\vee"] \\ \overline{\mathbb{O}}_{min} & \ni X \end{tikzcd}$$

clear that the rank of X is $= 1$

The most basic predictions of mirror symmetry:

$$\begin{tikzcd} & & \mathfrak{h}_X := H^2(X,\mathbb{C}) \\ Y \arrow[r] & X \arrow[ur, squiggly] \arrow[dr, squiggly] & \\ & & \mathfrak{t}_X := \mathrm{Lie}\, T_X \end{tikzcd}$$

max torus

$$T_X \subset \mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(X) =: G_X$$

Rmk: $G_X \leftarrow$ linear algebraic group

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Why: $A := \mathbb{C}[X]$, $A = \bigoplus_{i \geq 0} A_i$, $\dim A_i < \infty$

$\exists N \geq 0$ s.t. $A_{\leq N} := \bigoplus_{0 \leq i \leq N} A_i$ generates the whole A under multiplication.

$$G_X \hookrightarrow \mathrm{Aut}^{\mathbb{C}^\times}(A) \hookrightarrow GL(A_{\leq N}) \quad \checkmark$$

predictions:

$$\mathfrak{h}_X \simeq \mathfrak{t}_{X^\vee}$$

$$\mathfrak{t}_X \simeq \mathfrak{h}_{X^\vee}$$

$$\underset{\nwarrow}{X^{T_X}} \simeq \underset{\nearrow}{(X^\vee)^{T_{X^\vee}}}$$

two finite sets

In our example:

$$\mathfrak{h}_X = H^2(\tilde{S}_e) = H^2(\mathcal{B}_e) = \langle [C_i] \mid i=1,\dots,n-1 \rangle$$

$$T^\vee \curvearrowright T^*\mathbb{P}^{n-1} = Y^\vee, \quad \underset{\nwarrow\, \text{max}}{T^\vee} \subset PGL_n = \mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(X^\vee)$$

$$\mathrm{Lie}\, T^\vee = \langle \overset{\epsilon_i - \epsilon_{i+1}}{\overset{\shortparallel}{\alpha_i}}, \; i = 1,\dots,n-1 \rangle$$

$\nwarrow$ simple coroots for PGL_n

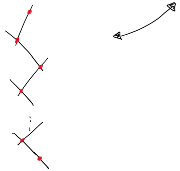
$$\begin{tikzcd} H^2(\tilde{S}_e) \arrow[r, phantom, "\simeq"] & \mathrm{Lie}\, T^\vee \\ {[C_i]} \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \alpha_i \arrow[u, phantom, "\in" sloped] \end{tikzcd}$$

$$\mathfrak{h}_{X^\vee} = H^2(T^*\mathbb{P}^{n-1}, \mathbb{C}) = \mathbb{C}$$

$T_X = T_e$ ← also 1-dim

$$(T^*\mathbb{P}^{n-1})^{T^\vee} \overset{Q}{=} (\mathbb{P}^{n-1})^{T^\vee} = \{ [0:\dots:\underset{i}{1}:0:\dots:0] \mid i = 1,\dots,n \}$$

$$\mathcal{B}_e^{T_e} =$$



Important: in general, even if X admits a symplectic resolution $Y \to X$, the dual $Y^\vee$ might not have a symplectic resolution.

prediction (quite unexpectable...)

$X^\vee$ ← has symplectic resolution $\iff$ $X^{T_X} = \{p_X\}$

unique $\mathbb{C}^\times$-fixed point to which X contracts via $\mathbb{C}^\times$-action

Assume that X admits a symplectic resolution $Y \to X$

Prop. $X^{T_X} = \{p_X\} \iff Y^{T_X}$ ← Finite

$\Leftarrow$ ← will only prove this direction today

1) $Y^{T_X} \to X^{T_X}$ is surjective:

pick $x \in X^{T_X}$, then $T_X \curvearrowright \pi^{-1}(x)$

proper

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$\pi^{-1}(x)^{T_X} \neq \emptyset$ ← take any generic $\upsilon : \mathbb{C}^\times \to T_X$

s.t. $Y^{T_X} = Y^{\upsilon(\mathbb{C}^\times)}$,

pick any $y \in \pi^{-1}(x)$

$\lim\limits_{t \to 0} \upsilon(t) y$ ← exists (as $\pi^{-1}(x)$-proper)

is $\upsilon(\mathbb{C}^\times)$-fixed

2) Y^{T_x} - finite so 1) $\Rightarrow$ X^{T_x} - finite

$\mathbb{C}^\times \curvearrowright X^{T_x}$ contracts to p_x

$\Downarrow$

X^{T_x} must be $= \{p_x\}$

Ex: $Y = \tilde{S}_e \to S_e = X$

← no conditions on e!

Question: when $S_e^{T_x} = \{e\}$?

(recall that this should correspond to the fact that $X^\vee$ has symplectic resol.)

$$\mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(X) = Z_G(e,h,f)$$

← this is true in most of the cases

$T_x = T_e$ ← max torus in $Z_G(e,h,f)$

← remember we had it already in subregular example

Set $M := Z_{T_e}(G)$ ← Levi subgroup of G

$\mathfrak{m} := \mathrm{Lie}\, M$

Ex: e - subregular $\Rightarrow$ $M = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right)$

Note: $e, h, f \in \mathfrak{m}$

Claim: $S_e^{T_e} = \{e\}$ iff $e \in \mathfrak{m} \leftarrow$ regular

prf: $M = \{ g \in G \mid tgt^{-1} = g \quad \forall t \in T_e \}$

$$\Downarrow$$

$$\mathfrak{m} = \{ x \in \mathfrak{g} \mid Ad_t(x) = x \quad \forall t \in T_e \}$$

$$\|$$

$$\mathfrak{g}^{T_e}$$

So $\underset{\uparrow}{\mathcal{N}(\mathfrak{m})} = \mathcal{N}(\mathfrak{g})^{T_e}$

nilpotent cone for $\mathfrak{m}$

We see that:

$$S_e^{T_e} = S_e \cap \mathcal{N}(\ell) = (e + \mathfrak{z}_{\mathfrak{g}}(f)) \cap \mathcal{N}(\mathfrak{m}) = S_e(\mathfrak{m})$$

$\dim S_e(\mathfrak{m}) = \dim \mathcal{N}(\mathfrak{m}) - \dim \mathbb{O}_e(\mathfrak{m}) \leftarrow$ is zero iff

$e \leftarrow$ regular in $\mathfrak{m}$

Prop: assume $\mathfrak{g} = \mathfrak{sl}_n$, then any e is regular in the corresponding Levi $\mathfrak{m}$

only in notes

prf. consider $\underset{\mathfrak{sl}_2}{\underset{\parallel}{\langle e,f,h\rangle}} \hookrightarrow \mathfrak{sl}_n \curvearrowright \mathbb{C}^n$

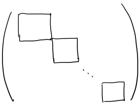
decompose $\mathbb{C}^n = \bigoplus_i V_i^{a_i} \qquad (V_i = S^i\mathbb{C}^2)$

e has a_i Jordan blocks of size $i+1$

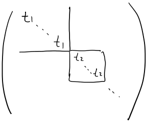
$$Z_{GL_n}(e,f,h) = \mathrm{Aut}_{\mathfrak{sl}_2}(\mathbb{C}^n) = \prod_i GL_{a_i}$$

$T_e = \prod_i T_{a_i}$ ← maximal torus in GL_{a_i}

Explicitly: if $e =$

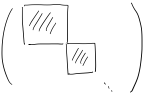


$\Rightarrow T_e =$



regular → $\mathfrak{m}$ ⇙

$M =$



Claim: in general, e will not be regular in $\mathfrak{m}$.

Ex: $\mathfrak{g} = DE$, $e \leadsto$ subreg.

Then $X = S_e = \mathbb{A}^2/\Gamma$

We claim that $G_X = \mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(X)$ is finite

This would imply $T_X = \{1\} \Rightarrow M = G$ and e is NOT regular in $\mathfrak{m}$.

prf. $\overset{\text{that } G_X\text{-finite}}{\vee}$ $X^{reg} := (\mathbb{A}^2 \setminus \{0\})/\Gamma$ ← smooth symplectic

↳ action of Γ is free on $\mathbb{A}^2 \setminus \{0\}$

$$G_{X^{reg}} = \mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(X^{reg}) \supset G_X$$

$$\mathbb{A}^2 \setminus \{0\} \twoheadrightarrow X^{reg}$$

↖ symplectic étale cover

↑ standard symplectic form here

Every symplectic $\mathbb{C}^\times$-equivariant automorphism of $\mathbb{A}^2 \setminus \{0\}$ induces Poisson $\mathbb{C}^\times$-equivariant automorphism of $\mathbb{C}[\mathbb{A}^2 \setminus \{0\}] = \mathbb{C}[x,y]$

It preserves $(\langle x,y \rangle, \{,\})$

↖ 2-dim vector space

So, $\mathrm{Aut}^{\mathbb{C}^\times}_{\{,\}}(\mathbb{A}^2 \setminus \{0\}) = SL_2$

goal: $G^\circ_{X^{reg}} = \{1\} \iff \mathrm{Lie}\, G_{X^{reg}} = \{0\}$

indeed $\mathrm{Lie}\, G_{X^{reg}} \hookrightarrow (\mathrm{Lie}\, SL_2)^\Gamma = \mathrm{Lie}\, Z_{SL_2}(\Gamma)$

pick $s \in \Gamma$ any element $\neq \pm 1$, it must be

Q

semi-simple $\Rightarrow$ can assume $s = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$

$Z_{SL_2}(s) = \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \middle| t \in \mathbb{C}^\times \right\}$ ⟜ 1-dim

so if $\mathrm{Lie}\, Z_{SL_2}(\Gamma) \neq \{0\} \Rightarrow Z_{SL_2}(\Gamma) = \mathbb{C}^\times$

$\Rightarrow \Gamma \subset Z_{SL_2}(\mathbb{C}^\times) = \mathbb{C}^\times$ must be $\mathbb{Z}/n\mathbb{Z}$!

But we assumed $\mathfrak{g} = DE$

Next time: assume that e is regular in $\mathfrak{m}$, describe the dual to S_e in this case.

Langlands dual

Spoiler:

$$\tilde{S}_e \to S_e \longleftrightarrow \underset{\substack{\parallel \\ Y^{\vee}}}{T^*(G^{\vee}/P^{\vee})} \to \underset{\substack{\parallel \\ X^{\vee}}}{\operatorname{Spec}\mathbb{C}[T^*(G^{\vee}/P^{\vee})]}$$

P-parabolic with Levi L

Will see that things are nice in type A and become much more complicated/interesting in other types.

closure of some nilpotent orbit

Spoiler: $X^{\vee} = \overline{\mathbb{O}}$ ← in type A

in general

$X^{\vee} = \operatorname{Spec}\mathbb{C}[\tilde{\mathbb{O}}]$ cover of nilp. orbit

Will then discuss parabolic Slodowy varieties. Duality becomes quite natural from this perspective.