Lecture 6

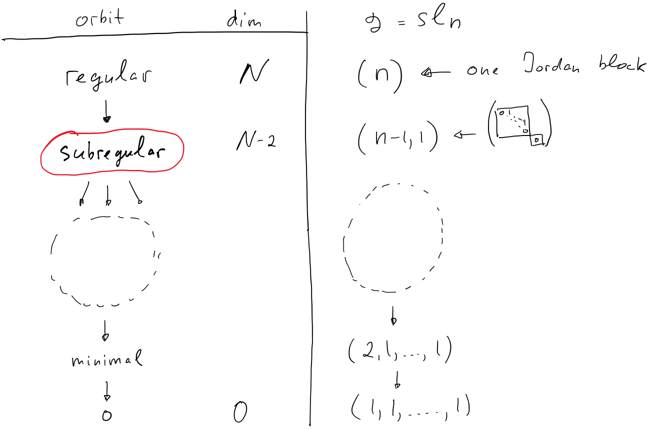
четверг, 13 февраля 2025 г. 08:14

Kleinian singularities

$\mathfrak{g} \leftarrow$ simple $/\mathbb{C}$, $N = \dim\mathfrak{g} - \dim\mathfrak{h} = \dim\widetilde{\mathcal{N}}$

$\mathbb{O}, \mathbb{O}' \subset \mathcal{N} \leftarrow$ nilpotent orbits, then $\mathbb{O}' \leq \mathbb{O}$

if $\mathbb{O}' \subset \overline{\mathbb{O}}$



orbit	dim	$\mathfrak{g} = \mathfrak{sl}_n$
regular	N	$(n) \leftarrow$ one Jordan block
↓		
subregular	$N-2$	$(n-1,1) \leftarrow$
↓ ↓ ↓		
		↓
↓		
minimal		$(2,1,\dots,1)$
↓		↓
0	0	$(1,1,\dots,1)$

See that for $e \leftarrow$ subregular, $\dim S_e = \underset{2}{\underset{\|}{\dim\mathcal{N} - \dim\mathbb{O}_e}}$

$\Downarrow$

$S_e \leftarrow$ singular surface

Moreover recall $S_e \cap \mathbb{O}' \neq \emptyset \Rightarrow \mathbb{O}_e \leq \mathbb{O}'$

$S_e \cap \mathbb{O}_e = \{e\}$

We conclude that $S_e = \{e\} \cup \underbrace{S_e \cap \mathbb{O}_{\mathrm{reg}}}_{\text{open, smooth, symplectic}}$

So $S_e \leftarrow$ surface with singularity at **one** point

Prop. [McKay corresp.] $\exists$ bijection between finite subgroups $\Gamma \subset SL_2(\mathbb{C})$ and ADE Dynkin diagrams

Ex: $\mathbb{Z}/_n\mathbb{Z} \hookrightarrow SL_2(\mathbb{C}) \longleftrightarrow A_{n-1}\ (\mathfrak{sl}_n)$

$$\begin{tikzcd} \mathbb{Z}/_n\mathbb{Z} \\ {[k]} \arrow[u, phantom, "\in" sloped] \arrow[r, mapsto] & \begin{pmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & e^{-\frac{2\pi i k}{n}} \end{pmatrix} \end{tikzcd}$$

$$\begin{tikzcd} Q_8 \arrow[r, hook] & SL_2(\mathbb{C}) \arrow[r, leftrightarrow] & D_4\ (\mathfrak{so}_8) \\ \text{quaternion group: } \{\pm 1, \pm i, \pm j, \pm k\} \arrow[u] \end{tikzcd}$$

$$\{\pm 1, \pm i, \pm j, \pm k\} \circlearrowright \mathbb{H} = \mathbb{C}^2$$

Thm [Slodowy] let $\Gamma \subset SL_2$ be finite sbgrp.

and let $\mathfrak{g}$ be the corresponding simple Lie alg.

Then $$\mathbb{A}^2/\Gamma \simeq S_e \,,$$ where e a subregular nilpotent in $\mathfrak{g}$

isomorphism of Poisson varieties

$\operatorname{Spec} \mathbb{C}[x,y]^{\Gamma}$

Q

We will discuss how this works for $\mathfrak{sl}_n$

Subregular slices for $\mathfrak{sl}_n$

$$e = e_{\alpha_1} + \dots + e_{\alpha_{n-1}} = \left(\begin{array}{c|c} \underbrace{\begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & & 1 \\ & & & 0 \end{matrix}}_{n-1} & \\ \hline & 0 \end{array}\right)$$

$$h = \left(\begin{array}{c|c} \begin{matrix} n-2 & & & \\ & n-4 & & \\ & & \ddots & \\ & & & 2-n \end{matrix} & \\ \hline & 0 \end{array}\right), \quad f = \left(\begin{array}{c|c} \begin{matrix} 0 & & & & \\ (n-2)\cdot 1 & 0 & & & \\ & (n-3)\cdot 2 & \ddots & & \\ & & \ddots & \ddots & \\ & (n-k-1)k & & 1\cdot(n-2) & 0 \end{matrix} & \\ \hline & 0 \end{array}\right)$$

$\uparrow$

Exercise to check that they indeed form an $\mathfrak{sl}_2$ - triple

Ex: $n=4$

$$h = \begin{pmatrix} 2 & & & \\ & 0 & & \\ & & -2 & \\ & & & 0 \end{pmatrix}, \quad f = \left(\begin{array}{ccc|c} 0 & & & \\ 2 & 0 & & \\ 0 & 2 & 0 & \\ \hline & & & 0 \end{array}\right)$$

$$S_e^{\mathfrak{g}} = e + \ker(\mathrm{ad}_f) \; - ?$$

$$\begin{array}{cccccccc} \mathfrak{sl}_n = & \mathfrak{sl}_{n-1} & \oplus & \mathbb{C}^{n-1} & \oplus & (\mathbb{C}^{n-1})^* & \oplus & \mathbb{C} & \text{as } \mathfrak{sl}_{n-1}\text{- module} \\ & \parallel & & \parallel & & \parallel & & \parallel & \\ & \underset{\substack{\downarrow \\ e}}{V_2} \oplus \underset{\substack{\downarrow \\ e^2}}{V_4} \oplus \dots \oplus \underset{\substack{\downarrow \\ e^{n-2}}}{V_{2n-4}} & & V_{n-2} & & V_{n-2} & & V_0 & \text{as } \mathfrak{sl}_2\text{-module} \end{array}$$

highest weight vectors $\Downarrow$

$$S_e^{\mathfrak{g}} = e + a_1 f + a_2 f^2 + \dots + a_{n-2} f^{n-2} + b \underset{\substack{\parallel \\ e_{\alpha_n}}}{E_{n-1,n}} + c \underset{\substack{\parallel \\ e_{-\theta}}}{E_{n,1}}$$

$$+ \; a \cdot \left(\frac{1}{n-1} \cdot \left(\sum_{i=1}^{n-1} E_{ii} - E_{nn} \right) \right)$$

Ex:

- $\mathfrak{sl}_2$

$$S_e^{\mathfrak{g}} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$$

- $\mathfrak{sl}_3$ $\quad S_e^{\mathfrak{g}} = \left\{ \begin{pmatrix} a/2 & 1 & 0 \\ a_1 & a/2 & b \\ c & 0 & -a \end{pmatrix} \right.$

- $\mathfrak{sl}_4$ $\quad S_e^{\mathfrak{g}} = \left\{ \begin{pmatrix} a/3 & 1 & 0 & 0 \\ 2a_1 & a/3 & 1 & 0 \\ 4a_2 & 2a_1 & a/3 & b \\ c & 0 & 0 & -a \end{pmatrix} \right\}$

Kazhdan grading:

$\deg a_i = 2i+2$

$\deg a = 2$

$\deg b = \deg c = n$ ← use that $e_{-\alpha_n}$, $e_{-\theta}$ are lowest vectors in V_{n-2}

Recall $S_e = S_e \cap \mathcal{N}$ and

$$\mathcal{N} = \{ X \in \mathfrak{sl}_n \mid \underbrace{\mathrm{Tr}(X^{l+1})}_{K_l} = 0 \quad \forall\, l = 1, \dots, n-1 \}$$

$$\begin{tikzcd}
X \arrow[r, mapsto] \arrow[d, phantom, "\cap"] & (K_1(X), \dots, K_{n-1}(X)) \\
S_e^{\mathfrak{g}} \arrow[r, "\varphi"] & \mathfrak{g}/\!/G \\
S_e \arrow[u, phantom, "\cup"] \arrow[r] & \{0\} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$\deg K_l = 2l+2$

Ex:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \qquad K_1 = 2a^2 + 2bc$$

$$\begin{pmatrix} a/2 & 1 & 0 \\ a_1 & a/2 & b \\ c & 0 & -a \end{pmatrix} \qquad \begin{aligned} K_1 &= \frac{3}{2}a^2 + 2a_1 \\ K_2 &= -\frac{3}{4}a^3 + 3aa_1 + 3bc \end{aligned}$$

$$\begin{pmatrix} a/3 & 1 & 0 & 0 \\ 2a_1 & a/3 & 1 & 0 \\ 4a_2 & 2a_1 & a/3 & b \\ c & 0 & 0 & -a \end{pmatrix} \qquad \begin{aligned} K_1 &= \frac{4}{3}a^2 + 8\overset{\deg = 4}{a_1} \\ K_2 &= -\frac{8}{9}a^3 + 8aa_1 + 12\overset{\deg = 6}{a_2} \\ K_3 &= \frac{28}{27}a^4 + \frac{16}{3}a^2a_1 + 32a_1^2 + 16aa_2 + 4bc \end{aligned}$$

linear! by degree observations

In general: $K_\ell = \underline{\# a_\ell} + P(a, a_1, \dots, a_{\ell-1})$, $\ell = 1, \dots, n-2$

$$K_{n-1} = bc + Q(a, a_1, \dots, a_{n-2})$$

$$\Downarrow$$

If we fix all values of K_ℓ then the coordinates on $\varphi^{-1}(K_1, \dots, K_{n-1})$ are a, b, c,

the only relation is:

$$bc = \# a^n + f_1(K_1, \dots, K_{n-1})\, a^{n-2} + \dots + f_n(K_1, \dots, K_{n-1})$$

$\deg = 2n$ $\nearrow$ $\quad$ $\uparrow$ $\quad$ s.t. $f_i(0,\dots,0) = 0$

some scalar,
can check that
with our normalizations $= -1$

$\swarrow\!\!\swarrow$

for $K_1 = \dots = K_{n-1} = 0$ get:

$$S_e \simeq \left\{ (a,b,c) \,\middle|\, a^n + bc = 0 \right\} = \mathbb{A}^2 / \mathbb{Z}/n\mathbb{Z}$$

$\deg a = 2, \ \deg b = n, \ \deg c = n$

So the $\mathbb{C}^\times$-action coincides with the $\mathbb{C}^\times$-action on $\mathbb{A}^2$
$(x,y) \mapsto (tx, ty)$

$$\begin{pmatrix} a \sim xy \\ b \sim x^n \\ y \sim y^n \end{pmatrix}$$

Poisson structure?

$A := \mathbb{C}[S_e]$

$\deg \{\,,\} = 2$ (from the definitions)

$\deg a = 2, \ \deg b = \deg c = n$

Consider $$T_e := \left(\begin{array}{cccc|c} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ \hline & & & & t \end{array} \right) \curvearrowright S_e$$

$\uparrow$

acts on S_e preserving the Poisson structure Q

T_e induces the grading on A that we will denote $\deg_{T_e}$

We have:

$\deg_{T_e}\{\ ,\ \} = 0$; $\deg_{T_e} a = 0$; $\deg_{T_e} b = -1$; $\deg_{T_e} c = 1$

Goal: check that $\{\ ,\ \}$ on S_e, $\mathbb{A}^2/\mathbb{Z}/_n\mathbb{Z}$ are the same.

$\{\ ,\ \} \leftarrow$ determined by $\{a,c\}$

$\deg\{a,c\} = n$; $\deg_{T_e}\{a,c\} = 1$

$$A_4 = \langle \underbrace{\underset{1}{c},\ \underset{-1}{b},\ \underset{0}{a^{\frac{n}{2}}}}_{T_e\text{ - weights}} \rangle$$

do not have this term for odd n

$\Downarrow$

$$\{a,c\} = \underline{k} \cdot c \qquad \mathbb{Q}$$

some nonzero constant (nonzero since otherwise $\{\ ,\ \} = 0$)

$\Downarrow$

After normalizing a if needed conclude that Poisson bracket that we get is the standard bracket on $\mathbb{A}^2/\mathbb{Z}/_n\mathbb{Z}$

Rem: can check $\underline{k=1}$! $\leftarrow$ use realization $S_e = N/\!/\!/_\chi Q$ from last time

(Exercise)