Lecture 5

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Last time

e, h, f $\rightsquigarrow$ $\mathfrak{sl}_2$-triple

$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ $\rightsquigarrow$ ad_h-weight decomposition

$C \subset \mathfrak{g}$ ad_h-invariant s.t.

$\mathfrak{g} = [\mathfrak{g}, e] \oplus C$ $\qquad \pi : \tilde{\mathcal{N}} \to \mathcal{N}$

$\mathcal{M}_e^{\mathfrak{g}} := e + C$; $\mathcal{M}_e := (e + C) \cap \mathcal{N}$; $\tilde{\mathcal{M}}_e = \pi^{-1}(\mathcal{M}_e)$

smooth, symplectic

Poisson

(are called MV slices/varieties)

Rmk. How to think about $C \subset \mathfrak{g}$.

Consider $\mathfrak{g}$ as an $\mathfrak{sl}_2 = \langle e, h, f \rangle$-module and decompose:

$$\mathfrak{g} \simeq \bigoplus V_i^{\oplus m_i}, \qquad V_i = S^i(\mathbb{C}^2)$$

For every V_i ; $V_i = e(V_i) \oplus \mathbb{C} v_i^{lw}$

lowest weight vector in V_i

Then C has a basis consisting of $v_i^{lw} + \mathrm{ad}_e \cdot (?)$

$$\mathfrak{z}_{\mathfrak{g}}(f) = \mathrm{Span}_{\mathbb{C}}(v_i^{lw})$$

$$\dim C = \dim \mathfrak{z}_{\mathfrak{g}}(f) = \sum_i m_i \overset{Q}{=} \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1$$

$$\chi := \langle e, - \rangle \in \mathfrak{g}^*$$

$(\,,\,)$ on $\mathfrak{g}_1$: $\quad (x,y) = \chi([x,y])$

nondegenerate (last time)

Pick $\ell \subset \mathfrak{g}_{-1}$ Lagrangian for $(\,,\,)$

$$\mathfrak{q} := \ell \oplus \bigoplus_{i \le -2} \mathfrak{g}_i \rightsquigarrow Q \subset G$$

$$(\mathfrak{q} = \mathrm{Lie}\, Q)$$

$Q \curvearrowright \mathfrak{g}$ ← Hamiltonian action with moment map:

$$\begin{tikzcd} \mu : \mathfrak{g} \arrow[r] & \mathfrak{q}^* \\ x \arrow[r, mapsto] & \langle x, - \rangle|_{\mathfrak{q}} \end{tikzcd}$$

Proved last time: $\chi|_{\mathfrak{q}} \in (\mathfrak{q}^*)^Q$ so can consider:

$$\mathfrak{g} /\!/\!/_{\chi} Q = \mu^{-1}(\chi) /\!/ Q$$

Thm 1 There exists an isomorphism of Poisson varieties: $\mathcal{M}_e^{\mathfrak{g}} \xrightarrow{\sim} \mu^{-1}(\chi)/\!\!/Q$

Cor. $\mathcal{M}_e^{\mathfrak{g}} \simeq S_e^{\mathfrak{g}}$ as Poisson varieties

We will deduce Thm 1 from the following Thm.

our main result for today ↙

Thm 2 The action map $Q \times \mathcal{M}_e^{\mathfrak{g}} \to \mathfrak{g}$ lands into $\mu^{-1}(\chi) \subset \mathfrak{g}$ and gives the Q-invariant isomorphism:

$$Q \times \mathcal{M}_e^{\mathfrak{g}} \xrightarrow{\sim} \mu^{-1}(\chi)$$

where the action of $Q \curvearrowright Q \times \mathcal{M}_e^{\mathfrak{g}}$ is $a \cdot (a', X) = (aa', X)$, where $a, a' \in Q$, $X \in \mathcal{M}_e^{\mathfrak{g}}$.

Thm 2 $\Rightarrow$
1) $Q \curvearrowright \mu^{-1}(\chi)$ is free
2) $\mathcal{M}_e^{\mathfrak{g}} \simeq \mu^{-1}(\chi)/Q$ (Thm 1)

Proof of Thm 2

① Image of $Q \times \mathcal{M}_e^{\mathfrak{g}}$ lies in $\mu^{-1}(\chi)$?

Recall $\mathcal{M}_e^{\mathfrak{g}} = e + C$; $C \subset \bigoplus_{i \le -1} \mathfrak{g}_i \overset{Q}{\subset} \mathfrak{q}^{\perp_{\mathfrak{g}}} = \{ x \in \mathfrak{g} \mid \langle x, \mathfrak{q} \rangle = 0 \}$

so if we take $a \in Q$, $y \in \mathfrak{q}$:

$$\langle Ad_a \cdot (e + C), y \rangle = \langle e + C, \overset{\overset{\mathfrak{q}}{\rotatebox{90}{$\in$}}}{Ad_{a^{-1}} \cdot y} \rangle =$$

$$= \langle e, Ad_{a^{-1}} \cdot y \rangle = \langle Ad_a \cdot e, y \rangle =$$

$$= (Ad_{a^{-1}} \cdot \chi)(y) = \chi(y).$$

② So we have a map:

$$Q \times \mathcal{M}_e^{\mathfrak{g}} \xrightarrow{\alpha} \mu^{-1}(\chi) = e + \mathfrak{q}^{\perp_{\mathfrak{g}}}$$

Define $\mathbb{C}^\times$-actions on both varieties by the following formulas:

our contracting action on $\mathcal{M}_e^{\mathfrak{g}}$

$$t.(a, y) = (t^{-h} a t^{h}, \, t^2 \cdot Ad_{t^{-h}}(y)), \quad a \in Q, \; y \in \mathcal{M}_e^{\mathfrak{g}}$$

$$t.z = t^2 \cdot Ad_{t^{-h}}(z), \quad z \in e + \mathfrak{q}^{\perp_{\mathfrak{g}}}$$

Claims: (a) α is $\mathbb{C}^\times$-equivariant

(b) $\mathbb{C}^\times$ contracts both of our varieties to a point

(c) $d\alpha_{(1,e)} : T_{(1,e)}(Q \times \mathcal{M}_e^{\mathfrak{g}}) \xrightarrow{\sim} T_e(e + \mathfrak{q}^{\perp_{\mathfrak{g}}})$ is an isomorphism

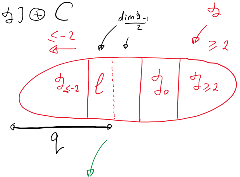
prf. (a) ← easy exercise

(b) Q <= $\mathfrak{q} \subset \bigoplus_{i \leq -1} \mathfrak{g}_i$; $\mathfrak{q}^{\perp_{\mathfrak{g}}} \subset \bigoplus_{i \leq 1} \mathfrak{g}_i$

(c)
$$\begin{tikzcd} d\alpha_{(1,e)}: \mathfrak{q} \oplus C \arrow[r] & \mathfrak{q}^{\perp_{\mathfrak{g}}} \\ \rotatebox{90}{$\in$} & \rotatebox{90}{$\in$} \\ (a,y) \arrow[r, mapsto] & {[e,a]+y} \end{tikzcd}$$

1) $\mathrm{ad}_e : \mathfrak{q} \overset{Q}{\xrightarrow{\sim}} [e, \mathfrak{q}]$ ← $\mathfrak{sl}_2$-rep. th-ry

2) $[e, \mathfrak{q}] \cap C = \{0\}$ <= $\mathfrak{g} = [e, \mathfrak{g}] \oplus C$



So, $d\alpha_{(1,e)}$ is an embedding.

Now, let's compare dimensions:

3) $\dim \mathfrak{q}^{\perp_{\mathfrak{g}}} = \dim \mathfrak{g} - \dim \mathfrak{q}$;

$$\dim \mathfrak{q} + \frac{\dim \mathfrak{g}_{-1}}{2} = \frac{\dim \mathfrak{g} - \dim \mathfrak{g}_0}{2}$$

$$\Downarrow$$

$$2 \dim \mathfrak{q} = \dim \mathfrak{g} - \dim \mathfrak{g}_0 - \dim \mathfrak{g}_{-1}$$

$$\Downarrow$$

$$\dim \mathfrak{g} = 2 \dim \mathfrak{q} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_{-1}$$

$$\begin{aligned} &\parallel \\ &\dim \mathfrak{q} + \dim \mathfrak{g}_0 + \dim \mathfrak{g}_1 \\ &\parallel \\ &\dim [e, \mathfrak{q}] + \dim C \end{aligned}$$

③ So, we have α s.t. (a), (b), (c) as above hold.

Now, we use the following general Lemma.

Lemma. Let $\alpha : X \to Z$ be a $\mathbb{C}^\times$-equivariant morphism of smooth affine $\mathbb{C}^\times$-varieties with contracting $\mathbb{C}^\times$-actions s.t. $d\alpha$ induces the isomorphism between thangent spaces at $\mathbb{C}^\times$-fixed points $x \in X$, $z \in Z$. Then α is isomorphism.

prf. $A = \mathbb{C}[X]$, $B = \mathbb{C}[Z] \curvearrowleft \mathbb{Z}_{\geq 0}$-graded

1) $\alpha \curvearrowleft$ isomorphism on tangent spaces at x, z
Let $\mathfrak{m}_x \subset A$, $\mathfrak{m}_z \subset B$ be the corresponding maximal ideals.

$$\hat{A}^{\mathfrak{m}_x} \simeq \mathbb{C}[[T_x X]] \; ; \quad \hat{B}^{\mathfrak{m}_z} = \mathbb{C}[[T_z Z]]$$

$$\Downarrow$$

α induces isomorphism of graded algebras $\alpha^*: \widehat{B}^{m_z} \xrightarrow{\sim} \widehat{A}^{m_x}$

2) $A \hookrightarrow \widehat{A}^{m_x}$ can be extracted from $\widehat{A}^{m_x}$ as follows:

Exercise $\quad A = \bigoplus_{i=0}^{\infty} \left(\widehat{A}^{m_x}\right)_i \quad$ (note that $\widehat{A}^{m_x} = \prod_{i=0}^{\infty} \left(\widehat{A}^{m_x}\right)_i$)

in particular we see that A must be isomorphic to the polynomial ring

3) It follows from 2) that α^* must induce the isomorphism $\alpha^*: B \xrightarrow{\sim} A$.

This finishes the proof of Thm 2

We get: $\quad Q \times \mathcal{M}_e^{\mathfrak{z}} \xrightarrow{\sim} H^{-1}(x) \Rightarrow \mathcal{M}_e^{\mathfrak{z}} \simeq \mathfrak{z} /\!/\!/_x Q$

Intersecting with $\mathcal{N} \subset \mathfrak{z}$ we get $\quad Q \times \mathcal{M}_e \xrightarrow{\sim} H_{\mathcal{N}}^{-1}(x)$

$$\Downarrow$$

$$\mathcal{M}_e \simeq \mathcal{N} /\!/\!/_x Q$$

Thm 3

$$Q \times \widetilde{\mathcal{M}}_e \xrightarrow{\sim} \mu_{\widetilde{\mathcal{N}}}^{-1}(\chi) \longleftarrow$$

$$\begin{tikzcd} \widetilde{\mathcal{N}} \arrow[r, "\pi"] \arrow[rr, bend right, "\mu_{\widetilde{\mathcal{N}}}"'] & \mathcal{N} \arrow[r, "\mu_{\mathcal{N}}"] & \mathfrak{q} \end{tikzcd}$$

$$\Downarrow$$

$$\widetilde{\mathcal{M}}_e \simeq \widetilde{\mathcal{N}} /\!/\!/_{\chi} Q$$

prf. Exercise (follows from Thm 2 by base change)

Poisson structures

How to compute the bracket on $S_e^{\mathfrak{g}}$

using the identification $S_e^{\mathfrak{g}} \xrightarrow{\sim} \mathfrak{g} /\!/\!/_{\chi} Q$

Start with $f_1, f_2 \in \mathbb{C}[S_e^{\mathfrak{g}}] \twoheadleftarrow \mathbb{C}[e + \mathfrak{q}^{\perp_{\mathfrak{g}}}]$

1) Find unique $\overset{Q\text{-invariant}}{\checkmark}$ lifts $g_1, g_2 \in \mathbb{C}[e + \mathfrak{q}^{\perp_{\mathfrak{g}}}]^{Q}$ of f_1, f_2

not so easy in practice!

2) Pick any $\tilde{g}_1, \tilde{g}_2 \in \mathbb{C}[\mathfrak{g}]$ lifting g_1, g_2

3) $\{f_1, f_2\} = \{\tilde{g}_1, \tilde{g}_2\}\big|_{S_e^{\mathfrak{g}}}$

Compatibility of Poisson structures is a consequence of the following exercise:

Exersice $\forall$ nilpotent orbit $\mathbb{O}$,

$$S_e \cap \mathbb{O} \simeq \mathbb{O} /\!\!/\!\!/_{\chi} Q$$

as symplectic varieties