Lecture 4

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Important comment

I want to spell out one thing from last time more carefully.

How to think about the Poisson structure on Se ←

1) $\mathbb{O}_{\mathrm{reg}} \subset \mathcal{N}$
symplectic

2) $\mathbb{O}_{\mathrm{reg}} \cap S_e$ – symplectic (restrict form from $\mathbb{O}_{\mathrm{reg}}$)

3) $\mathbb{C}[\mathbb{O}_{\mathrm{reg}} \cap S_e] = \mathbb{C}[S_e]$ ← use that S_e – normal and $S_e \cap \mathbb{O}'$ – symplectic ⇒ even-dim

⇓

Poisson structure on $\mathbb{C}[S_e]$ is induced by the symplectic structure on $\mathbb{O}_{\mathrm{reg}} \cap S_e$

$S_e \hookrightarrow \mathcal{N}$ ← Poisson

↑ has induced Poisson structure

Warning: $\mathbb{C}[\mathfrak{g}] \twoheadrightarrow \mathbb{C}[S_e^{\mathfrak{g}}]$ ← NOT a homomorphism of Poisson alg.

Ex. $Z \hookrightarrow X$ ← smooth symplectic $\omega \in \Lambda^2 T_X^*$

↑ smooth

Can happen that $\omega|_Z$ ← nondegenerate at every point.

Then Z has induced symplectic structure

but Z is not a Poisson subvariety.

$\mathbb{A}^2 \hookrightarrow \mathbb{A}^4$ $\quad \{x_i, y_j\} = \delta_{ij}$

x_1, y_1 $\quad$ x_1, y_1, x_2, y_2

⇓

$\mathbb{A}^2$ has induced symplectic structure $\{x_1, y_1\} = 1$

but $\mathbb{C}[\mathbb{A}^4] \twoheadrightarrow \mathbb{C}[\mathbb{A}^2]$ ← not Poisson

$x_2 \longmapsto 0$

$y_2 \longmapsto 0$

$\{x_2, y_2\} = 1 \longmapsto 1$

Last time

Q:

$$\begin{tikzcd} \text{symplectic?} \arrow[d] & & \\ \tilde{S}_e \arrow[r, hook] \arrow[d, "\text{Poisson?}"] & \tilde{\mathcal{N}} = T^*\mathcal{N} \arrow[d] \\ S_e \arrow[r, hook] & \mathcal{N} \end{tikzcd}$$

Claim (from last time)

$B_0 \subset G$ Borel subgroup, $\mathfrak{b}_0 := \operatorname{Lie} B_0$, $\mathfrak{n}_0 \subset \mathfrak{b}_0$

nilpotent radical

$$G \times^{B_0} \mathfrak{n}_0 := \{ (g, x) \in G \times \mathfrak{n}_0 \} / B_0 \; ; \quad (g,x) \cdot b = (gb, \mathrm{Ad}_b^{-1} \cdot x)$$

action is free

Exercise:

$$\begin{tikzcd} q : G \times^{B_0} \mathfrak{n}_0 \arrow[r] & \mathcal{B} \\ {[g, x]} \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & g B_0 g^{-1} \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

called associated bundle

vector bundle of rank $\dim \mathfrak{n}_0$

The following was also mentioned last time:

(a) There exists a G-equivariant isomorphism of vector bundles

$$\begin{tikzcd} T^*\mathcal{B} \arrow[rr, phantom, "\simeq"] \arrow[dr, "p"'] & & G \times^{B_0} \mathfrak{n}_0 \arrow[dl, "q"] \\ & \mathcal{B} & \end{tikzcd}$$

prf. Pick $B \in \mathcal{B} \Rightarrow \underset{\substack{\| \\ p^{-1}(B)}}{T^*_B \mathcal{B}} = (\mathfrak{g}/\mathfrak{b})^* = \mathfrak{b}^\perp = \mathfrak{n}$

Let us now describe the fiber of q over B

$B = h B_0 h^{-1}$ for some $h \in G$

h is determined uniquely up to multiplication by an element of B_0

$$q^{-1}(B) = \{ [g,x] \mid g \in hB_0,\ x \in \mathfrak{n}_0 \}$$

$$[g,x] \longmapsto g B_0 g^{-1} = h B_0 h^{-1} = B$$

Have canonical identification:

$$\begin{tikzcd} q^{-1}(B) \arrow[r, "\sim"] & \mathfrak{n} = p^{-1}(B) \\ {[g,x]} \arrow[r, mapsto] \arrow[u, phantom, "\in" rotate=90] & \mathrm{Ad}_g \cdot x \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

$\Downarrow$

obtain the desired isomorphism $T^*\mathcal{B} \simeq G \times^{B_0} \mathfrak{n}_0$

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argument is the same as above ✓

In general $T^*(G/P_0) \simeq G \times^{P_0} \mathfrak{p}_0^\perp$,

where $\mathfrak{p}_0^\perp = \{ x \in \mathfrak{g}^* \mid x|_{\mathfrak{p}_0} = 0 \}$

Claim from the last time

(b) The moment map for $G \curvearrowright T^*(G/P_0)$

is given by $\mu : G \times^{P_0} \mathfrak{p}_0^\perp \to \mathfrak{g}^*$

$\in \qquad\qquad \in$

$$[g,\eta] \longmapsto \mathrm{Ad}_g \cdot \eta$$

Cor.

$$\begin{tikzcd} \tilde{\mathcal N} = T^*\mathcal B \arrow[r, "\pi"] \arrow[rr, bend right] & \mathcal N \arrow[r, hook] & \mathfrak g \\ (x,b) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \arrow[rr, mapsto, bend right, "\varphi"'] & & x \arrow[u, phantom, "\rotatebox{90}{$\in$}"] \end{tikzcd}$$

is the moment map => Poisson

prf. identification $G \times^{B_0} \mathfrak n_0 \simeq T^*\mathcal B \simeq \{(x,b) \mid x \in \mathfrak n\}$ is precisely given by $[g,x] \longmapsto (\mathrm{Ad}_g(x),\ g b_0 g^{-1})$

So:

$$\begin{tikzcd} \tilde S_e \arrow[r, hook] \arrow[d] & \tilde{\mathcal N} \arrow[d, "\text{Poisson}"] \\ S_e \arrow[r, hook] & \mathcal N \arrow[l, bend left, "\text{induced Poisson}"] \end{tikzcd}$$

remains to check that symplectic form on $\tilde{\mathcal N}$ restricts to the symplectic form on $\tilde S_e$

Idea

$$\tilde{\mathcal N} = T^*(G/B_0) = T^*G /\!/\!/ B_0 = (G \times \mathfrak g^*) /\!/\!/ B_0$$

Then $G \times S_e^{\mathfrak g} \hookrightarrow G \times \mathfrak g^*$

symplectic form on $G \times \mathfrak g^*$ restricts to the symplectic form on $G \times S_e^{\mathfrak g}$ (explicit computation)

$\Swarrow$

$$S_e = (G \times S_e^{\mathfrak g}) /\!/\!/ B_0 \hookrightarrow (G \times \mathfrak g^*) /\!/\!/ B_0 = \tilde{\mathcal N}$$

symplectic form restricts to symplectic form

not detailed because we will discuss alternative approach (in some sense better)

Gan - Ginzburg's approach

Setting $e, h, f \rightsquigarrow \mathfrak{sl}_2$-triple

Def. Let $C \subset \mathfrak{g}$ be ad_h-invariant subspace s.t. $\mathfrak{g} = [\mathfrak{g}, e] \oplus C$. Then

$\mathcal{M}_e^{\mathfrak{g}} = e + C \hookrightarrow \mathfrak{g}$ is called MV slice

Mirkovic-Vybornov

Ex. $C = \mathfrak{z}_{\mathfrak{g}}(e) \Rightarrow \mathcal{M}_e^{\mathfrak{g}} = S_e^{\mathfrak{g}}$

Rmk (Exercise): everything that we proved for Slodowy slices also holds for MV slices!

Makes sense to talk about:

$\mathcal{M}_e = \mathcal{M}_e^{\mathfrak{g}} \cap \mathcal{N} \rightsquigarrow$ Poisson

$\tilde{\mathcal{M}}_e := \pi^{-1}(\mathcal{M}_e) \rightsquigarrow$ smooth, symplectic

Ex. same formula as in the Slodowy variety case defines *contracting* $\mathbb{C}^\times$-action on $\mathcal{M}_e^{\mathfrak{g}}$. Q

To see that, note that $h \curvearrowright C$ with non-positive weights: indeed if V - irrep. of $\mathfrak{sl}_2$ $\Rightarrow$ complement to $e(V)$ is the lowest weight vector.

Goal: realize $\mathcal{M}_e$, $\tilde{\mathcal{M}}_e$, $\mathcal{M}_e^{\mathfrak{g}}$ as Hamiltonian reductions of $\mathcal{N}, \tilde{\mathcal{N}}, \mathfrak{g}$.

As corollary will conclude:

$$S_e \simeq \mathcal{M}_e; \quad \tilde{S}_e \simeq \tilde{\mathcal{M}}_e, \quad S_e^{\mathfrak{g}} \simeq \mathcal{M}_e^{\mathfrak{g}}$$

isomorphisms of Poisson varieties!

identification will be very implicit

Construction

$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \leftarrow$ h-weight decomposition

Set $\chi := \langle e, - \rangle \in \mathfrak{g}^*$

Define $(\ ,\ )$ on $\mathfrak{g}_1$ by: $(x, y) = \chi([x, y])$

Lemma $(\, , \,)$ is non-degenerate skew-symmetric form.

Prf.

- $(x, y) = \langle e, [x, y] \rangle = \langle \mathrm{ad}_e(x), y \rangle$

Q

- $\langle \, , \, \rangle : \mathfrak{g}_1 \times \mathfrak{g}_{-1} \to \mathbb{C}$ $\rightsquigarrow$ nondegenerate
- $\mathrm{ad}_e : \mathfrak{g}_{-1} \xrightarrow{\sim} \mathfrak{g}_1$ is an isomorphism (rep. theory of $\mathfrak{sl}_2$)

Pick $\ell \subset \mathfrak{g}_{-1}$ Lagrangian for $(\, , \,)$

Define $\mathfrak{q} := \ell \oplus \bigoplus_{i \le -2} \mathfrak{g}_i$ $\rightsquigarrow$ nilpotent subalgebra

Ex: $$e = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \quad h = \begin{pmatrix} n-1 & & & 0 \\ & n-3 & & \\ & & \ddots & \\ 0 & & & -n+1 \end{pmatrix}$$

⇙ Q

$$\mathfrak{q} = \mathfrak{n}_- = \begin{pmatrix} 0 & & 0 \\ \ast & \ddots & \\ \ast & \ast & 0 \end{pmatrix}$$

Let $Q \subset G$ be a unipotent subgroup s.t. $\mathrm{Lie}\, Q = \mathfrak{q}$

The action $Q \curvearrowright \mathfrak{g}$ is Hamiltonian with moment map $\mu: \mathfrak{g} \to \mathfrak{q}^*$

Q $\qquad$ $\begin{array}{ccc} \mathfrak{g} & \to & \mathfrak{q}^* \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ x & \mapsto & \langle x, - \rangle|_{\mathfrak{q}} \end{array}$

Claim: $\chi|_{\mathfrak{q}} \in \mathfrak{q}^*$ is Q-invariant.

prf. need to check that $\mathrm{ad}_y(\chi|_{\mathfrak{q}}) = 0$ for $y \in \mathfrak{q}$

$$(\mathrm{ad}_y \cdot \chi)(\overset{z \in \mathfrak{q}}{z}) = \chi(-[y,z]) = -\langle e, [y,z] \rangle$$

Recall that $\mathfrak{q} \subset \bigoplus_{i \le -1} \mathfrak{g}_i$, $e \in \mathfrak{g}_2$,

so $\langle e, [y,z] \rangle$ can be nonzero only if

$y, z \in \mathfrak{q}_{-1}$. But then $y, z \in \mathfrak{l}$ $\Rightarrow$

$\Rightarrow \langle e, [y,z] \rangle = (y,z) = 0$

Thm 1 $\quad \mathcal{M}_e^{\mathfrak{g}} \simeq \mu^{-1}(\chi)/Q$

$\nearrow$

iso. of Poisson varieties

(note that the RHS does not depend on C!)

$\Uparrow$

Thm 2 $\quad Q \times \mathcal{M}_e^{\mathfrak{g}} \simeq \mu^{-1}(\chi)$

$\uparrow$

will prove next time