Lecture 3

понедельник, 3 февраля 2025 г. 07:00

Poisson structures (on $\mathcal{N}, S_e, \tilde{\mathcal{N}}, \tilde{S}_e$)

Last time:

(*)

$$\begin{tikzcd}
 & \text{smooth} \arrow[dl] \arrow[dr] & \\
\tilde{S}_e \arrow[rr, hook] \arrow[dd] & & \tilde{\mathcal{N}} = T^*(G/B) \arrow[dd] \\
 & \text{resol. of sing.} \arrow[l] \arrow[r] & \\
S_e \arrow[rr, hook] & & \mathcal{N} \\
 & \text{affine, normal} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Goal: introduce compatible Poisson / Symplectic structures on the varieties in (*).

①

We start with the Poisson structure on $\mathfrak{g}$.

Recall $\mathfrak{g} \overset{\langle\, ,\, \rangle}{\simeq} \mathfrak{g}^*$

Def. The Poisson bracket on $\mathbb{C}[\mathfrak{g}^*] = S^\bullet \mathfrak{g}$ is given by $\{x, y\} := [x, y]$ for $x, y \in \mathfrak{g}$

(and then uniquely extended to $S^\bullet \mathfrak{g}$

using Leibniz rule).

this definition makes sense for arbitrary Lie alg.

Prop. every orbit $\mathbb{O} \subset \mathfrak{g}^*$ is a symplectic subvariety of $\mathfrak{g}^*$

← it actually forms a symplectic leaf of $\mathfrak{g}$

prf. Pick $f \in \mathbb{O}$ $\Rightarrow$

$$\begin{tikzcd} & \mathfrak{g} \arrow[r] & T_f\mathbb{O} \\ \ker = \mathfrak{z}_{\mathfrak{g}}(f) \arrow[ur, bend left] \arrow[d, phantom, "\parallel"] & X \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \xi_X|_f \arrow[u, phantom, sloped, "\in"] \\ \mathfrak{g}_f & & \end{tikzcd}$$

vector field, corresponding to the infinitesimal action of X on $\mathbb{O}$

$\Downarrow$

$$T_f\mathbb{O} \simeq \mathfrak{g}/\mathfrak{g}_f$$

need: $(\mathfrak{g}/\mathfrak{g}_f) \times (\mathfrak{g}/\mathfrak{g}_f) \xrightarrow{\omega} \mathbb{C}$

$\Uparrow$ descends

consider: $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$: only depends on $x \in \mathfrak{g}/\mathfrak{g}_f$

$$(x, y) \mapsto f([x, y]) = (\mathrm{ad}_x \cdot f)(-y)$$

ω - nondegenerate:

$$(\mathrm{ad}_x \cdot f)(y) = 0 \quad \forall y \iff x \in \mathfrak{g}_f$$

—

remains to check:

1) $d\omega = 0$ ← exercise

2) ω ← compatible with Poisson structure on $\mathfrak{g}$

check 2): $x, y \in \mathfrak{g} \rightsquigarrow \xi_x, \xi_y$ ← vector fields on $\mathfrak{g}$

$$\{x,y\}^{\mathbb{O}}(f) = \omega(\xi_x, \xi_y)(f) = f([x,y]) = \{x,y\}^{\mathfrak{g}}(f)$$

So, we defined Poisson structure on $\mathfrak{g}$.

Claim $\mathcal{N} \subset \mathfrak{g}$ Poisson subvariety

prf.
$$\mathcal{N} = \mathbb{C}[\mathfrak{g}] \Big/ \big(\underbrace{\mathbb{C}[\mathfrak{g}]^G_+}_{= I}\big)$$

need to prove that this ideal is Poisson ← $\{\mathbb{C}[\mathfrak{g}], I\} \subset I$

$$\forall f \in \mathbb{C}[\mathfrak{g}]^G_+ ; \quad \{x, f\} = x.f = 0 \quad \forall x \in \mathfrak{g}$$

⇙

$$\{\mathbb{C}[\mathfrak{g}], \mathbb{C}[\mathfrak{g}]^G_+\} = 0 \overset{\text{Leibniz rule}}{\Longrightarrow} I\text{ - Poisson}$$

Let us now deal with Slodowy slices.

$$S_e^{\mathfrak{g}} = e + \mathfrak{z}_{\mathfrak{g}}(f) \hookrightarrow \mathfrak{g} = \bigcup \overset{\text{symplectic leaves}}{\mathbb{O}}$$

Prop. Poisson structure on $\mathfrak{g}$ induces the Poisson structure on $S_e^{\mathfrak{g}}$

prf.

Exercise let $\overset{\text{smooth}}{Z} \overset{\text{closed}}{\hookrightarrow} \overset{\text{smooth}}{X} \leftarrow$ Poisson s.t.

(i) Z is transversal to every symplectic leaf $\mathbb{O}$ of X

(ii) $\forall x \in Z$, $x \in \mathbb{O}$, the symplectic (nondegenerate) pairing on $T_x\mathbb{O}$ restricts to the nondegenerate pairing on $T_x\mathbb{O} \cap T_x Z = T_x(\mathbb{O} \cap Z)$

Then the Poisson structure on X induces the Poisson structure on Z

Apply to $Z = S_e^{\mathfrak{g}}$, $X = \mathfrak{g}$

(i) $\Leftarrow$ Lecture 2

(ii): $T_x(S_e^{\mathfrak{g}}) = \ker(\mathrm{ad}_f)$

$$T_x \mathbb{O} = \mathrm{im}(\mathrm{ad}_x)$$

$$\uparrow$$

$$([x,a] \times [x,b]) \longmapsto \langle x, [a,b] \rangle \quad (a, b \in \mathfrak{g})$$

Assume $y = [x,b] \in T_x \mathbb{O}$ s.t.

$$\omega(T_x \mathbb{O} \cap T_x S_e^{\mathfrak{g}}, y) = 0$$

$$\Updownarrow$$

$$\langle x, [a,b] \rangle = 0 \quad \forall a \in \mathfrak{g} \text{ s.t. } [x,a] \in \ker(\mathrm{ad}_f)$$

$$\parallel$$

$$\langle \underbrace{[x,a]}_{\substack{\cap \\ \ker(\mathrm{ad} f)}}, b \rangle \iff b \in (\mathrm{im}(\mathrm{ad}_x) \cap \ker(\mathrm{ad}_f))^{\perp_{\mathfrak{g}}}$$

orthogonal w.r.t. $\langle , \rangle$

$$\parallel$$

$$\mathrm{im}(\mathrm{ad}_x)^{\perp_{\mathfrak{g}}} + \ker(\mathrm{ad}_f)^{\perp_{\mathfrak{g}}}$$

$$\parallel$$

$$\ker(\mathrm{ad}_x) + \mathrm{im}(\mathrm{ad}_f)$$

$$\Swarrow$$

$$\underline{y = [x,b] \in [x, [f, \mathfrak{g}]]}$$

remains: $\forall x \in S_e^{\mathfrak{g}}$:

$$[x,[f,\mathfrak{g}]] \cap \ker(\mathrm{ad}_f) = 0 \qquad (*)$$

When $x = e \Rightarrow$

$$[e,\mathfrak{g}] \oplus \ker(\mathrm{ad}_f) \overset{\star}{=} \mathfrak{g} \overset{\star}{=} [f,\mathfrak{g}] \oplus \ker(\mathrm{ad}_e)$$

$\star \Rightarrow [e,\mathfrak{g}] \cap \ker(\mathrm{ad}_f) = 0$, so $(*)$ holds for $x=e$

moreover: $\mathrm{ad}_e : [f,\mathfrak{g}] \xrightarrow{\sim} [e,[f,\mathfrak{g}]]$

$\nwarrow$ from $\star$

$\Downarrow$

$\mathrm{ad}_x : [f,\mathfrak{g}] \xrightarrow{\sim} [x,[f,\mathfrak{g}]]$

$\nwarrow$ bijective for x close to e

$\Downarrow$

$\dim [x,[f,\mathfrak{g}]]$ – fixed (for x close to e)

$\Downarrow$

Then $(*)$ must hold for x close to e.

Using contracting $\mathbb{C}^\times$-action we deduce that it holds for any x.

Rmk. see that $S_e^{\mathfrak{g}} \cap \mathbb{O} \leftarrow$ symplectic in $S_e^{\mathfrak{g}}$

Where we are: $\mathcal{N}$ given by zeroes of functions that lie in Poisson center $\swarrow \mathfrak{Z}$

$$S_e^{\mathfrak{g}} \hookrightarrow \mathfrak{g} \quad \Rightarrow \quad S_e \hookrightarrow \mathcal{N}$$

$\rotatebox{90}{$\in$}$ $\nearrow$ Poisson $\qquad$ $\rotatebox{90}{$\in$}$ Poisson

② Hamiltonian actions and Hamiltonian reductions

M ← smooth, $\overset{\text{can drop}}{\text{affine}}$, Poisson variety

K ← alg. grp.

Assume $K \curvearrowright M$ preserves $\{\,,\,\}$.

Def. The action $K \curvearrowright M$ ← Hamiltonian

if $\exists$ a $\overset{K\text{-invariant}}{\vee}$ (moment) map $\mu : M \to \mathfrak{k}^*$ s.t.:

$$(*) \quad \{\mu^*(x), f\} = \xi_x(f) \qquad \forall x \in \mathfrak{k},\ f \in \mathbb{C}[M]$$

Can consider: ← called Hamiltonian

$$\begin{tikzcd} \mathfrak{k} \arrow[r, "H"] & \mathbb{C}[M] \\ x \arrow[r, mapsto] \arrow[u, phantom, "\in" description] & H_x := \mu^*(x) \arrow[u, phantom, "\in" description] \end{tikzcd}$$

Claim

(a) H ← homom. of Lie alg.

(b) μ ← Poisson

prf. $\{H_x, H_y\} = \xi_x(H_y) \overset{\text{use } K\text{-inv}}{=} H_{[x,y]}$

Ex $M = T^*X$, $\omega = d\lambda$

$K \curvearrowright X \rightsquigarrow K \curvearrowright M$

$$\begin{tikzcd}
\mathfrak{k} \arrow[r, "H"] & \mathrm{Vect}(X) \arrow[r, hook] & \mathbb{C}[M] \\
x \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & \xi_x \arrow[u, phantom, sloped, "\in"] &
\end{tikzcd}$$

note $\{\xi_x, \xi_y\} = [\xi_x, \xi_y] = \xi_{[x,y]}$

$\Downarrow$

H is indeed a homom. of Lie alg.

Ex. $M = T^*(K/P)$, $P \subset K$ subgroup

discussed similar statement last time for G/B, same argument works in general

Then

$$\begin{tikzcd}
M \arrow[dd, "\mu"] & \overset{K\text{-equiv}}{=} & K \times^P \mathfrak{p}^\perp = (K \times \mathfrak{p}^\perp)/P \\
 & & {[k,x]} \arrow[u, phantom, sloped, "\in"] \arrow[d, mapsto] \\
\mathfrak{k}^* & \ni & \mathrm{Ad}_k(x)
\end{tikzcd}$$

$(k,x)\cdot p = (kp, p^{-1}x)$

Exercise

Def. Hamiltonian reduction

$$M/\!\!/\!\!/K := \mu^{-1}(0)/\!\!/K = \operatorname{Spec} \mathbb{C}[\mu^{-1}(0)]^K$$

Claim: $\{\,,\,\}$ on $\mathbb{C}[M]$ induces the Poisson structure on $$\Big(\mathbb{C}[M]/\underbrace{\mathbb{C}[M]\,\mu^*(\mathfrak{m}_0)}_{I}\Big)^K$$

prf: enough to check that:

if $f \in I$ and

$g \in \mathbb{C}[M]$ is s.t. $[g] \in \big(\mathbb{C}[M]/I\big)^K$

$\Downarrow$

$\{f,g\} \in I$

this condition implies that $\{\mu^*(x), g\} = \xi_x(g) \in I$

decompose: $f = \sum_i a_i \cdot \mu^*(x_i)$, $x_i \in \mathfrak{m}_0$, $a_i \in \mathbb{C}[M]$

$\Downarrow$

$$\{f,g\} = \sum_i \underset{\substack{\cap \\ \mathbb{C}[M]\,\mu^*(\mathfrak{m}_0) \\ \| \\ I}}{\{a_i,g\}\,\mu^*(x_i)} + \sum_i a_i \cdot \underset{\substack{\cap \\ I}}{\{\mu^*(x_i), g\}}$$

Rem if $M \leftarrow$ symplectic, $K \overset{\text{free}}{\curvearrowright} \mu^{-1}(0) \leftarrow$ smooth s.t.

$$\mu^{-1}(0) \xrightarrow{} \mu^{-1}(0) /\!/ K$$

$\uparrow$ principal K-bundle

$\Longleftarrow$

$$\mu^{-1}(0) /\!/\!/ K \leftarrow \text{symplectic}$$

Ex. $M = T^*X$, $K \overset{\text{free}}{\curvearrowright} X$ s.t. $X /\!/ K$ - smooth

Then $(T^*X) /\!/\!/ K = T^*(X /\!/ K)$

$\nearrow$

How to think about this:

For $p \in X$, $T_{[p]}(X/K) = T_pX / \mathrm{Span}(\xi_x \mid x \in k)$

$\nearrow$ exactly $\mu^*(x)$

So $T(X/K) =$

$$\begin{tikzcd} \cdots T_pX/\mathrm{Span}(\xi_x) \cdots \arrow[d] \\ X/K \end{tikzcd}$$

Back to our story:

$$\{(x,\mathfrak{b}) \mid x \in \mathfrak{n}\} \simeq \underset{\in}{G \times^{B_0}} \mathfrak{n}_0$$

$$= \{(\mathrm{Ad}_g(x), \mathrm{Ad}_g(\mathfrak{b}_0))\} \leftarrow\!\shortmid (g,x)$$

$$\begin{tikzcd}
\tilde{S}_e \arrow[r, hook] \arrow[d, "\pi"] & T^*(G/B_0) \arrow[d, "\pi"] \arrow[dr, "\mu"] & \\
S_e \arrow[r, hook] & \mathcal{N} \arrow[r, hook] & \mathfrak{g}
\end{tikzcd}$$

$\mu \leftarrow$ Poisson $\Rightarrow \pi$ - Poisson

Remains to define the symplectic structure on $\tilde{S}_e$

Enough to check the following claim:

Claim

$$\tilde{S}_e \hookrightarrow T^*(G/B)$$

a symplectic subvariety

Claim implies that $\tilde{S}_e \to S_e \leftarrow$ Poisson morphism

(use that $\tilde{\mathcal{N}} \to \mathcal{N}$ is)

Let us explain the proof of the claim.

$$\underset{\circlearrowleft}{T^*(G/B)} = TG /\!\!/\!\!/ B = \underset{\circlearrowleft}{(G \times \mathfrak{g}^*)} /\!\!/\!\!/ B$$

$$\tilde{S}_e = (\,?\,)/\!\!/\!\!/B = (G \times S_e^{\mathfrak{b}})/\!\!/\!\!/B$$

Need to check:

(1) $G \times S_e^{\mathfrak{b}} \overset{\text{symplectic}}{\hookrightarrow} G \times \mathfrak{b}^*$

will discuss next time

(2) $\tilde{S}_e = (G \times S_e^{\mathfrak{b}})/\!\!/\!\!/B$