Lecture 23

суббота, 26 апреля 2025 г. 11:09

Recall:

$e, h, f \in \mathfrak{g} \rightsquigarrow \mathfrak{sl}_2$-triple

$C \subset \mathfrak{g} \rightsquigarrow \mathrm{ad}_h$-invariant, $\mathfrak{g} = [\mathfrak{g}, e] \oplus C$

$\mathcal{M}_e^{\mathfrak{g}} = e + C$, $\mathcal{M}_e = \mathcal{M}_e^{\mathfrak{g}} \cap \mathcal{N}$, $\widetilde{\mathcal{M}}_e \rightsquigarrow$ preimage in $\widetilde{\mathcal{N}}$

proved (see Lecture 4):

$$\mathcal{M}_e^{\mathfrak{g}} \simeq S_e^{\mathfrak{g}}; \quad \mathcal{M}_e \simeq S_e; \quad \widetilde{\mathcal{M}}_e \simeq \widetilde{S}_e$$

same argument: $\widetilde{\mathcal{M}}(e, \mathfrak{p}) \simeq \widetilde{S}(e, \mathfrak{p}) \rightsquigarrow$ preimages under $T^*(G/P) \to \mathcal{N}$

Goal for today:

all coeff. are ≥ 0

$$G = GL_n, \quad \lambda = \underset{\substack{\| \\ (a_1, \ldots, a_m)}}{\omega_{k_1} + \ldots + \omega_{k_m}}, \quad \mu \leq \lambda$$

Will construct:

corresponding to certain particular $C \subset \mathfrak{gl}_{k_1 + \ldots + k_m}$

$$\begin{tikzcd}
\widetilde{W}^{\underline{\lambda}}_H \arrow[r, phantom, "\simeq"] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{\mathcal{M}}(e, \mathfrak{p}) \arrow[r, phantom, "\overset{\text{know}}{\simeq}"] \arrow[d] & \widetilde{S}(e, \mathfrak{p}) \arrow[d] \\
\overline{W}^{\lambda}_H \arrow[r, phantom, "\simeq"] & \mathcal{M}(e, \mathfrak{p}) \arrow[r, phantom, "\overset{\text{know}}{\simeq}"] & S(e, \mathfrak{p})
\end{tikzcd}$$

Gan-Ginzburg

$(*)$

$e, \mathfrak{p} \leftrightarrow \lambda, H$

Start with $\widetilde{\mathcal{W}}^{\lambda}_{\mu}$.

Recall

$$\begin{tikzcd}
\widetilde{Gr}^{\lambda}_{GL_n} = \left\{ \underbrace{\overset{k_m}{\overbrace{L_{-m} \subset L_{-m+1}}}}_{\cdot z} \subset \dots \subset \underbrace{\overset{k_1}{\overbrace{L_{-1} \subset L_0}}}_{\cdot z} \;\middle|\; \begin{array}{l} \dim L_{1-i}/L_{-i} = k_i \\ z(L_{-i}) \subset L_{-i-1} \end{array} \right\} \arrow[d] \arrow[d, bend right=60, "\text{see Lecture 21}"'] \\
\overline{Gr}^{\lambda}_{GL_n} = \left\{ L_{-m} \subset L_0 \;\middle|\; \begin{array}{l} \dim L_0/L_{-m} = k_1 + \dots + k_m \\ \text{JNF of } z \text{ is } \leq \begin{pmatrix} \overbrace{\begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{smallmatrix}}^{a_1} & & \\ & \overbrace{\begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{smallmatrix}}^{a_2} & \\ & & \ddots \end{pmatrix} =: e_\lambda \end{array} \right\}
\end{tikzcd}$$

lattices

Now, $\widetilde{\mathcal{W}}^{\lambda}_{\mu} \leftarrow$ preimage of $\mathbb{W}_\mu \cap \overline{Gr}^{\lambda}$.

$\mu = (b_1, \dots, b_n)$.

Recall:

$$\mathbb{W}_\mu = GL_n[[z^{-1}]]_1 \cdot z^\mu = z^\mu \cdot \left(z^{-\mu} GL_n[[z^{-1}]]_1 z^\mu \right)$$

recall that this one is iso to $GL_n[[z^{-1}]]_1$

Claim: (1) $\mathbb{W}_\mu \subset \left(z^\mu GL_n[[z^{-1}]]_1 \right) \cdot 1$

$$\shortparallel$$

$$z^\mu \cdot \mathbb{W}_0 \subset Gr_G$$

(2) $\mathbb{W}_\mu = \left\{ z^\mu \cdot (g_{ij}) \in z^\mu GL_n[[z^{-1}]]_1 \;\middle|\; g_{ij} \in \delta_{ij} + z^{b_j - b_i - 1} \mathbb{C}[[z^{-1}]] \right\}$

prf. (1) $\mathbb{W}_\mu = U[[z^{-1}]]_1 \, T[[z^{-1}]]_1 \, z^\mu \, U_-[[z^{-1}]]_1$

//

$$z^{\mu}\cdot\Big(\underbrace{z^{-\mu}\,U[[z^{-1}]]_1\,z^{\mu}}_{\substack{\cap\\ U[[z^{-1}]]_1\\ \cap\\ z^{\mu}GL_n[[z^{-1}]]_1}}\cdot T[[z^{-1}]]_1\,U_-[[z^{-1}]]_1\Big)$$

$$(2)\quad \mathbb{W}_{\mu} = z^{\mu}\cdot\left(z^{-\mu}\,GL_n[[z^{-1}]]_1\,z^{\mu}\right)$$

$$z^{-\mu}(x_{ij})\,z^{\mu} = \left(z^{b_j-b_i}\,x_{ij}\right)$$

$$\begin{array}{c}\| \\ g_{ij} \in \delta_{ij} + z^{b_j-b_i-1}\,\mathbb{C}[[z^{-1}]]\end{array}$$

more detailed argument: $z^{-\mu}\,U[[z^{-1}]]_1\,z^{\mu}$ $\ni$ consists of $\left(z^{b_j-b_i}x_{ij}\right)$

then we mult. by $B_-[[z^{-1}]]_1$

pole of these remains the same

$$\text{Set}\quad E_{\mu} := \{\underset{\substack{\uparrow\\ \text{standard basis in } \mathbb{C}^n}}{e_i},\, z e_i, \dots, z^{b_i-1}e_i \mid i=1,\dots,n\}$$

Lemma:

open condition! ($\mathbb{W}_0$ - open)

$$\overline{Gr}^{\lambda} \cap z^{\mu}\mathbb{W}_0 = \left\{ L \subset L_0 \in \overline{Gr}^{\lambda} \;\middle|\; \begin{array}{l}\text{image of } E_{\mu}\\ \text{gives a basis in } L_0/L\end{array}\right\}$$

$$\cup$$

prf. Assume $L \subset L_0$, $L = z^{\mu}g(L_0)$, $g \in G[z^{-1}]_1$.

1) $g(L_0) \subset z^{-\mu}L_0 \Rightarrow g(e_j) = e_j + \underbrace{\mathrm{Span}_{\mathbb{C}}\left(z^{-a}e_i \mid a = -1,\dots,-b_i\right)}_{z^{-\mu}E_{\mu}.}$ $(*)$

$$\Downarrow$$

$$g(L_0) + z^{-\mu}E_{\mu} = z^{-\mu}L_0 \iff L + E_{\mu} = L_0$$

2) $\dim L_0/L = \dim E_\mu \Rightarrow E_\mu$ gives a basis in L_0/L ✓

similar argument

3) In the opposite direction, if E_μ gives a basis in L_0/L then $z^{-\mu}L + z^{-\mu}E_\mu = L_0 \Rightarrow$ every e_i is of the form (*)
so we can find basis in $z^{-\mu}L$ of the form $x_i = e_i - \dots$ lin. comb of elements of E_μ

The matrix $g = (x_1, x_2, \dots, x_n)$ lies in $GL_n[z^{-1}]_{\underline{1}} \Rightarrow z^{-\mu}L \in \mathbb{W}_0 \Rightarrow$
$\Rightarrow L \in z^{\mu}\mathbb{W}_0$

Obtain embeddings:

$$\{0\} \overset{k_m}{\subset} F_1 \subset \dots \overset{k_1}{\subset} F_m$$

$$\begin{tikzcd}
\widetilde{W}^{\underline{\lambda}}_{\mu} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & T^*\mathcal{F}_{\underline{\lambda}} \arrow[d] \\
\overline{W}^{\lambda}_{\mu} \arrow[r, hook] & \overline{\mathbb{O}}_{e_\lambda}
\end{tikzcd}$$

cartesian from the def. (**)

via basis E_μ $\qquad$ $k_1 + \dots + k_m$

$$(L_{-m} \subset L_{-m+1} \subset \dots \subset L_0) \mapsto (\{0\} \subset L_{-m+1}/L_{-m} \subset \dots \subset L_0/L_{-m} = \mathbb{C}^N)$$

$\cdot z$ $\quad$ $\cdot z$ $\quad$ $\cdot z$ our nilpotent operator

In order to finish the proof of (*) remains to identify image of $\overline{W}^{\lambda}_{\mu}$ in $\overline{\mathbb{O}}_{e_\lambda} = \overline{\mathbb{O}}_{p}$ with $\mathcal{M}(e_\mu, p)$ for appropriate nilpotent e_μ and $\mathcal{C} \subset \mathfrak{gl}_N$

(the identification $\widetilde{W}^{\underline{\lambda}}_{\mu} \simeq \widetilde{\mathcal{M}}(e_\mu, p)$ follow using that (**) is cartesian)

if $(L \subset L_0) \in \overline{\mathcal{W}}^{\lambda}_{H}$, then:

$$\begin{tikzcd} \cdot z \curvearrowright & L_0/L \arrow[r, phantom, "\simeq"] & \mathbb{C}^N \\ & {[z^{b-1}e_j]} \arrow[u, phantom, sloped, "\in"] & e_j^{(b)} \arrow[l, mapsto] \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

$j = 1, \dots, n$
$b = 1, \dots, b_j$

given by some matrix X, let X_{ij} be its block

$$\begin{array}{c|c} & e_j^{(1)} \; \cdots \; e_j^{(b_j)} \\ \hline \begin{matrix} e_i^{(1)} \\ \vdots \\ e_i^{(b_i)} \end{matrix} & \underbrace{\left(\begin{matrix} \ddots & & \\ & \ddots & \\ & & \ddots \end{matrix} \right)}_{b_j} \left.\vphantom{\begin{matrix} 1 \\ 1 \\ 1 \end{matrix}}\right\} b_i \end{array}$$

We have: $e_j^{(b)} = [z^{b-1}e_j] \mapsto [z^b e_j]$ basis vector if $b \neq b_j$

The only interesting case:

$[z^{b_j-1}e_j] \mapsto [z^{b_j}e_j]$.

Now, $L = (z^H g) L_0$, $g = (g_{ij})$,

$$g_{ij} = \delta_{ij} + z^{b_j - b_i - 1} \cdot \sum_{k=0}^{\boxed{b_j-1}} p_{ij}^{(k)} z^{-k}$$

$z^H g e_j \in L_0 \Rightarrow$ see that $p_{ij}^{(k)} = 0$ for $k > b_{j}-1$

$$g \cdot e_j = \sum_i g_{ij} e_i = e_j + \sum_{i,k} z^{b_j - b_i - 1} p_{ij}^{(k)} z^{-k} e_i$$

$$\Downarrow$$

$$\underbrace{z^H g e_j}_{L \ni} - z^{b_j} e_j = \sum_{i,k} z^{b_j - k - 1} p_{ij}^{(k)} e_i$$

$$\Downarrow$$

$$[z^{b_j} e_j] \overset{\text{mod } L}{=} \sum_{i,k} \left(-p_{ij}^{(k)}\right) \cdot \overset{e_i^{(b_j-k)}}{\overset{\parallel}{[z^{b_j-k-1} e_i]}}$$

$$X_{ij} = \begin{pmatrix} 0 & 0 & & 0 & -p_{ij}^{(b_j-1)} \\ \delta_{ij} & \ddots & & \vdots & \vdots \\ & \ddots & \ddots & 0 & -p_{ij}^{(0)} \\ & & \ddots & \vdots & 0 \\ 0 & & & \delta_{ij} & 0 \end{pmatrix}$$

Define:

$$C \subset \mathfrak{gl}_N \qquad \left\{ X \in \mathfrak{gl}_N \;\middle|\; X_{ij} = \underbrace{\begin{pmatrix} 0 & 0 & & \boxed{\ast} \\ \delta_{ij} & 0 & 0 & \vdots \\ & \ddots & \ddots & 0 \\ 0 & & \delta_{ij} & 0 \end{pmatrix}}_{} \Big\} b_j \right\}$$

$$e_\mu := X \quad \text{s.t.} \quad X_{ij} = \begin{pmatrix} 0 & & 0 \\ \delta_{ij} & \ddots & \\ & \delta_{ij} & 0 \end{pmatrix}.$$

note that it's even simpler that $\mathfrak{z}_{\mathfrak{gl}_N}(f_\mu)$

Ex: $\mu = (3, 2) \Rightarrow$

$$X = \left[\begin{array}{ccc|cc} 0 & 0 & \ast & 0 & \ast \\ 1 & 0 & \ast & 0 & \ast \\ 0 & 1 & \ast & 0 & 0 \\ \hline 0 & 0 & \ast & 0 & \ast \\ 0 & 0 & \ast & 1 & \ast \end{array}\right]$$

Claim: (a) C ⟜ ad_h-invariant

(b) $[\mathfrak{gl}_N, e_\mu] \oplus C = \mathfrak{gl}_N$

prf (a) $\leftarrow$ clear

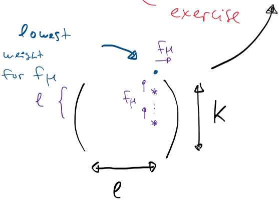
(b) consider the $\mathfrak{sl}_2$-triple e_μ, h_μ, f_μ.

Decompose $\mathbb{C}^N = \bigoplus M_k \otimes L_k$ $\hookleftarrow$ k-dim. irred. of $\mathfrak{sl}_2$

Then $\mathcal{H}_N = \mathrm{End}(\mathbb{C}^N) = \bigoplus \mathrm{Hom}(M_k, M_\ell) \otimes L_k^* \otimes L_\ell$

$$C = \bigoplus \mathrm{Hom}(M_k, M_\ell) \otimes \ker(f_\mu^\ell, L_k^*) \otimes \ker(f_\mu, L_\ell)$$

exercise



iso to L_k

(already $\mathfrak{sl}_2$ computation)

Remains to check that for $L_k \otimes L_\ell$ we have:

$$\left(\ker(f^\ell, L_k) \otimes \ker(f, L_\ell)\right) \overset{(*)}{} \oplus \mathrm{im}(e, L_k \otimes L_\ell) = L_k \otimes L_\ell$$

1) easy to see that dimensions are complement

2) (*) generates the whole $L_k \otimes L_\ell$ under the action of e: ← Exercise

e via $\cdot(x \otimes 1 + 1 \otimes x)$

$$\underbrace{\langle 1 \otimes 1, x \otimes 1, \dots, x^{\ell-1} \otimes 1 \rangle}_{(*)} \subset \langle 1, \dots, x^{k-1} \rangle \otimes \langle 1, \dots, x^{\ell-1} \rangle$$

$\times$ ← means that we can cancel it out

$$1 \otimes 1 \overset{e}{\longmapsto} \cancel{x \otimes 1} + 1 \otimes x$$

$$x \otimes 1 \longmapsto \cancel{x^2 \otimes 1} + x \otimes x$$

$$1 \otimes x \longmapsto \cancel{x \otimes x} + 1 \otimes x^2$$

$$X^2 \otimes 1 \longmapsto \cancel{X^3 \otimes 1} + X^2 \otimes X$$

$$X \otimes X \longmapsto \cancel{X^2 \otimes X} + X \otimes X^2$$

$$\vdots$$

will not explain

1) + 2) $\Rightarrow$ ✓

Hikita - Nakajima

universal quantization of $\overline{W}^{\lambda}_{H} = S(d,p) = X$

$Y^{\vee}$

$\parallel$

$$C_u\left(Y^{\lambda}_{H}\right) \simeq H^{*}_{T^{\vee} \times \mathbb{C}^{\times}_{\hbar}}\left(\tilde{S}(p^{\vee}, a^{\vee})\right)$$

quotient of $Y_H \supset \mathbb{C}[A_i^{(s)}]$

commutative subalgebra

have tautological bundles $\mathcal{V}_i$ on $T^{*}(G^{\vee}/P^{\vee})$, restrict to bundles on $\tilde{S}(p^{\vee}, a^{\vee})$

Main diagram:

use that $E_i^{()}, F_i^{()}$ have nonzero u-degree

classical result of Spaltenstein

$$\begin{tikzcd}
 & \mathbb{C}[A_i^{(s)}] \arrow[dl] \arrow[dr, mapsto] \arrow[dr, two heads] \arrow[dd, dash, "(*)"'] \arrow[dd, dash, "(**)"] & \\
C_u\left(Y^{\lambda}_{H}\right) \arrow[d, "(a)"'] & & C_s(\mathcal{V}_i) \in H^{*}_{T^{\vee} \times \mathbb{C}^{\times}_{\hbar}}\left(\tilde{S}(p^{\vee}, a^{\vee})\right) \arrow[d, "(b)"] \\
 & &
\end{tikzcd}$$

$$\begin{tikzcd} C_\upsilon(\tilde{Y}^\lambda_H) \arrow[d, equal] & & H^*_{T^\vee\times\mathbb{C}^\times_\hbar}\left(\tilde{S}(p^\vee,a^\vee)^{T^\vee}\right) \arrow[d, equal] \\ \bigoplus_{p\in(\tilde{W}^\lambda_H)^T} \mathbb{C}[\mathfrak{h},\hbar] \arrow[rr, equal] & & \bigoplus_{p\in\tilde{S}(p^\vee,a^\vee)^{T^\vee}} \mathbb{C}[\mathfrak{t}^\vee,\hbar] \end{tikzcd}$$

(a) $\rightsquigarrow$ embedding, to see that enough to check that $\dim C_\upsilon(\mathbb{C}[\overline{W}^\lambda_H]) \leq \#(\tilde{W}^\lambda_H)^T$, this can be checked using parabolic Slodowy variety realization and elementary combinatorics of symmetric functions

(b) $\rightsquigarrow$ embedding, to see that enough to check that $H^*_{T_{X^\vee}\times\mathbb{C}^\times_\hbar}(Y^\vee)$ is free over $H^*_{T_{X^\vee}\times\mathbb{C}^\times_\hbar}(pt)$.

Enough to construct a finite $T_{X^\vee}\times\mathbb{C}^\times_\hbar$-equivariant affine paving of $Y^\vee$. Consider cocharacter $t\mapsto(\upsilon(t),t^d)$, $d\gg 0$, get $\mathbb{C}^\times \curvearrowright Y^\vee$ $\rightsquigarrow$ contracts to $(Y^\vee)^{T_{X^\vee}\times\mathbb{C}^\times_\hbar}$ - finite.

Then BB decomposition gives the desired affine paving.

$(*)$ $\rightsquigarrow$ highest weight of Verma module, compute by reducing to λ $\rightsquigarrow$ minuscule case via comult.

$(**)$ $\rightsquigarrow$ restriction of $C_s(U_i)$ to fixed points, compute by reducing to $T^*(\check{G}/\check{P})$ - case (then standard and easy)

Rmk: if you compute things carefully, will see that our identification sends $\hbar\mapsto 2\hbar$ $\rightsquigarrow$ this is very important from the perspective of BVLS duality