Lecture 22

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Last time:

$$\left(\widetilde{W}^{\underline{\lambda}}_{\mu}\right)^T \leftrightarrow \left\{ z^{\underline{\mu}} \;\middle|\; \begin{array}{l} \underline{\mu} = (\mu_1, \dots, \mu_m) \\ \mu_i \in W\mu_i \\ \mu_1 + \dots + \mu_n = \mu \end{array} \right\}$$

symmetric def. of $\widetilde{W}^{\underline{\lambda}}_{\mu}$: fix $\mu = \mu_- + \mu_+$

$$\widetilde{W}^{\underline{\lambda}}_{\mu} \simeq \widetilde{W}^{\underline{\lambda}}_{\mu_-, \mu_+} := \left\{ \begin{array}{l} (\mathcal{T}_0^{B_-}, \mathcal{T}_0, \mathcal{T}_1, \dots, \mathcal{T}_m, \mathcal{T}_m^{B}) \\ \sigma_i : \mathcal{T}_i|_{\mathbb{P}^1 \setminus \{0\}} \simeq \mathcal{T}_{i+1}|_{\mathbb{P}^1 \setminus \{0\}} \\ s = s_0 \text{ — triv of } \mathcal{T}_0|_\infty \;\}_{\deg \le \lambda_i} \\ \deg \mathcal{T}_0^{B_-} = w_0\mu_- \,;\; \deg \mathcal{T}_m^{B} = w_0\mu_+ \\ \mathcal{T}_0^{B_-}|_\infty = B \,;\; \mathcal{T}_m^{B}|_\infty = B_- \end{array} \right.$$

Hecke transform by $-w_0\mu_-\cdot\infty$ at ∞.

Cor: for any $\underline{\lambda}, \lambda'$; $\mu \le \lambda$, $\mu' \le \lambda'$

there exist open embeddings

$$\begin{tikzcd}
\widetilde{W}^{\underline{\lambda}}_{\mu} \times \widetilde{W}^{\underline{\lambda}'}_{\mu'} \arrow[r, hook, "\widetilde{m}"] \arrow[d] & \widetilde{W}^{(\underline{\lambda}, \underline{\lambda}')}_{\mu+\mu'} \arrow[d] \\
\overline{W}^{\lambda}_{\mu} \times \overline{W}^{\lambda'}_{\mu'} \arrow[r, "m"] & W^{\lambda+\lambda'}_{\mu+\mu'}
\end{tikzcd}$$

and birational maps s.t. diagram is commutative

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prf.

$$\widetilde{W}^{\underline{\lambda}}_{\mu} \simeq \widetilde{W}^{\underline{\lambda}}_{\mu,0} \;;\; \widetilde{W}^{\underline{\lambda}'}_{\mu'} \simeq \widetilde{W}^{\underline{\lambda}'}_{0,\mu'}$$

$$\begin{tikzcd}
(\mathcal{T}_0^{B_-}, \mathcal{T}_0, \dots, \mathcal{T}_m \simeq \mathcal{T}^{triv}) \times (\mathcal{T}^{triv} = \mathcal{T}_0', \mathcal{T}_1', \dots, \mathcal{T}_{m'}', \mathcal{T}_{m'}'^{,B}) \arrow[d, mapsto] \\
(\mathcal{T}_0^{B_-}, \mathcal{T}_0, \dots, \mathcal{T}_{m-1}, \mathcal{T}^{triv}, \mathcal{T}_1', \dots, \mathcal{T}_{m'}', \mathcal{T}_{m'}'^{,B_-})
\end{tikzcd}$$

$$\widetilde{W}^{(\underline{\lambda}, \underline{\lambda}')}_{\mu, \mu'} \simeq \widetilde{W}^{(\underline{\lambda}, \underline{\lambda}')}_{\mu+\mu'}$$

image in $\widetilde{W}^{(\underline{\lambda}, \underline{\lambda}')}_{\mu+\mu'}$ consists of $(\dots, \mathcal{T}_m, \dots)$

s.t. $\mathcal{T}_m \simeq \mathcal{T}^{triv}$ ← this is an open condition!

Example:

$$\overline{W}^{(N,0)}_{(N-k,k)} = \left\{ M \in \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} x_{22} = z^k + \dots \\ \deg x_{21}, x_{12} < k \\ \det M = z^N \end{array} \right\}$$

$$\overline{\mathcal{W}}^{(N,0)}_{(N-k,k)} \times \overline{\mathcal{W}}^{(N',0)}_{(N'-k',k')} \xrightarrow{m} \overline{\mathcal{W}}^{(N+N',0)}_{(N+N'-k-k',k+k')}$$

$$\overset{\overset{M_1}{\shortparallel}}{\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}} \cdot \overset{\overset{M_2}{\shortparallel}}{\begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}} = \begin{pmatrix} A' & B' \\ C' & D \end{pmatrix}$$

$B' \to B$

divide with a remainder by D

$C' \to C$

$$A = \frac{z^{k_1+k_2} + BC}{D}$$

Then $m(M_1, M_2) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

Ex: if $k \geq N \Rightarrow$ multiplication morphism is a product (μ a antidominant in this case)

no need to divide with remainder

Rem: "matrix" description above generalizes to arbitrary $\overline{\mathcal{W}}^{\lambda}_{\mu} = U_1[[z^{-1}]]\, T[[z^{-1}]]_1\, z^{\mu}\, U_-[[z^{-1}]] \cap \overline{G[z] z^{\lambda} G[z]}$

multiplication can be described in these terms

Rem: multiplication lifts to: $\mathcal{W}_{\mu} \times \mathcal{W}_{\mu'} \to \mathcal{W}_{\mu+\mu'}$

(quantizes to comultiplication $Y_{\mu+\mu'} \to Y_{\mu} \otimes Y_{\mu'}$)

How multiplication helps:

Assume: μ a dominant

Recall: $$\left(\widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}\right)^{T} = \left\{ z^{\underline{\mu}} \;\middle|\; \begin{array}{l} \mu_i \in W\lambda_i \\ \sum_i \mu_i = \mu \end{array} \right\}$$

$\Downarrow$

Fix a decomposition $\underline{\mu} = (\mu_1, \dots, \mu_m)$, get

T-equivariant open embedding:

map is Poisson

$$\mathcal{W}^{\lambda_1}_{\mu_1} \times \dots \times \mathcal{W}^{\lambda_m}_{\mu_m} \hookrightarrow \widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}$$

image $\mathbb{O}_{\underline{\mu}}$ contains unique T-fixed point $z^{\underline{\mu}}$

Thm:

(a) $\widetilde{W}^{\underline{\lambda}}_{\mu} = \bigcup_{\underline{\mu}} \mathbb{O}_{\underline{\mu}}$ ← get Darboux coordinates on $\widetilde{W}^{\underline{\lambda}}_{\mu}$

(b) $\mathbb{O}_{\underline{\mu}} \simeq \prod T^* \mathbb{A}^{\langle \rho, \lambda_i - \mu_i \rangle}$

iso of symplectic var

(c) $Y^{\lambda_i}_{\mu_i} \simeq D(\mathbb{A}^{\langle \rho, \lambda_i - \mu_i \rangle})$

quantization of $\overline{W}^{\lambda}_{\mu}$ with generic parameter

(d) Verma modules for Y^{λ}_{μ} can be explicitly constructed as follows:

$$Y^{\lambda}_{\mu} \to Y^{\lambda_1}_{\mu_1} \otimes \dots \otimes Y^{\lambda_m}_{\mu_m} \simeq D(\mathbb{A}^{\langle \rho, \lambda_1 - \mu_1 \rangle}) \otimes \dots \otimes D(\mathbb{A}^{\langle \rho, \lambda_m - \mu_m \rangle})$$

$$\curvearrowright$$

$$\bigotimes \mathbb{C}[\mathbb{A}^{\langle \rho, \lambda_i - \mu_i \rangle}]$$

$\nearrow$

in particular, to understand characters and highest weights of these modules, it is enough to deal with the case $\lambda = \lambda_i$ ← is minuscule! So, $W^{\lambda}_{\mu} \simeq T^{*}\mathbb{A}^{\langle \rho, \lambda - \mu \rangle}$.

So, multiplication morphisms provide "building blocks" for our slices. Will allow us to deal with Hikita-Nakajima next time.

Note: next time we will identify $\widetilde{W}^{\underline{\lambda}}_{\mu}$ (in type A) with parabolic Slodowy var.

I do not know how to see decomposition (b) in these terms.

proof (of Thm)

(a) $\mathbb{C}^{\times} \curvearrowright \overline{W}^{\lambda}_{\mu}$ ← contracts to z^{μ}

loop rotation action

Lifts to $\mathbb{C}^{\times} \curvearrowright \widetilde{W}^{\lambda}_{\mu}$.

Morphism $\widetilde{W}^{\underline{\lambda}}_{\mu} \xrightarrow{\pi} \overline{W}^{\lambda}_{\mu}$ is projective.

Consider $\mathbb{C}^{\times} \xrightarrow{\eta} \mathbb{C}^{\times} \times T \curvearrowright \widetilde{W}^{\underline{\lambda}}_{\mu}$

$t \longmapsto (t^d, 2\rho(t))$

For $d \gg 0$, we have:

(1) $\left(\widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}\right)^{\eta(\mathbb{C}^\times)} = \left(\widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}\right)^{\mathbb{C}^\times \times T} = \{ z^{\underline{\mu}} \}$

(2) $\mathbb{C}^\times \overset{\eta}{\curvearrowright} \overline{\mathcal{W}}^{\lambda}_{\mu}$ contracts it to $\{z^\mu\}$

(3) $\mathbb{C}^\times \overset{\eta}{\curvearrowright} \widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}$ contracts it to fixed points

(use (2) + that π ← proper)

Pick $x \in \widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}$, $\lim_{t\to\infty} \eta(t).x = z^{\underline{\mu}}$, for some $\underline{\mu}$

Goal: check that $x \in \mathbb{O}_{\underline{\mu}}$.

comes from appropriate cocharacters of $T \times \mathbb{C}^\times \curvearrowright \mathcal{W}^{\lambda_i}_{\mu_i}$

Idea: $\exists$ action of $\mathbb{C}^\times \curvearrowright \prod \mathcal{W}^{\lambda_i}_{\mu_i} = \mathbb{O}_{\underline{\mu}}$

s.t. $m : \mathbb{O}_{\underline{\mu}} \hookrightarrow \widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}$ is $\mathbb{C}^\times$-equiv.

How this action looks like:

$(x_1, \dots, x_m) \mapsto (t x_1, \mu_1(t^{-1}) t x_2, \dots, (\mu_1 + \dots + \mu_{k-1})(t^{-1}) t x_k, \dots, (\mu_1 + \dots + \mu_{m-1})(t^{-1}) t x_m)$

Now: $\exists t$ s.t. $t.x \in \mathbb{O}_{\mu}$ (since $\mathbb{O}_\mu$ is open neighb. of $z^{\underline{\mu}}$)

$\Downarrow$

$x \in \mathbb{O}_\mu$ (use $\mathbb{C}^\times$-equiv of $\mathbb{O}_{\underline{\mu}} \hookrightarrow \widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu}$)

(b) ← need to describe $\mathcal{W}^{\lambda_i}_{\mu_i}$, λ_i ← minuscule, $\mu_i \in W\lambda_i$

So, from now on set $\lambda = \lambda_i$ ← minuscule

$\mu = \mu_i \in W\lambda$

not dominant!

Have a map $\underset{\substack{\circlearrowleft \\ \mathbb{C}^\times}}{\mathcal{W}^{\lambda}_{\mu}} \longrightarrow \underset{\substack{\circlearrowleft \\ \mathbb{C}^\times}}{\mathrm{Gr}^{\lambda}} = G/P_\lambda$ ← acts trivially!

loop

$R^{\lambda}_{\mu} := \{ x \in \overline{\mathcal{W}}^{\lambda}_{\mu} \mid \lim_{t\to\infty} 2\rho(t) \cdot x = z^{\mu} \}$

repellent in Gr^{λ} to z^{μ}

Claim 1: $R^{\lambda}_{\mu} \xrightarrow{\sim} T^{\lambda}_{\mu}$ ← this statement is true for any λ, μ

prf: $W^{\lambda}_{\mu} = U[[z^{-1}]]_1 \, \underset{\circlearrowright \text{ T - conjugation}}{T[[z^{-1}]]_1} \, z^{\mu} U_-[[z^{-1}]]_1 \cap (G[z] z^{\lambda} G[z])$

⇙

$R^{\lambda}_{\mu} = \underbrace{T[z^{-1}]_1}_{\{1\}''} z^{\mu} U_-[z^{-1}]_1 \cap G[z] z^{\lambda} G[z]$

$T^{\lambda}_{\mu} = U_-(K) \cdot z^{\mu} \cap Gr^{\lambda}_{G} \subset U_-(K) z^{\mu} / U_-(\mathcal{O}) =$
$= z^{\mu} \cdot U_-(K)/U_-(\mathcal{O})$

We have $U_-[[z]] \times U_-[z^{-1}]_1 \xrightarrow{\sim} U_-(K) \Rightarrow$

$\exists \xi : U_-[z^{-1}]_1 \xrightarrow{\sim} U_-(K)/U_-(\mathcal{O})$

Pick $[u_- z^{\mu}] \in T^{\lambda}_{\mu}$, $u_- z^{\mu} = z^{\mu} \tilde{u}_-$,

$\tilde{u}_- \in U_-(K)/U(\mathcal{O}) \overset{\xi^{-1}}{\simeq} U_-[z^{-1}]_1$

Get a map
$$\begin{tikzcd} T^{\lambda}_{\mu} \arrow[r] & R^{\lambda}_{\mu} \\ {[u_- z^{\mu}]} \arrow[r, mapsto] \arrow[u, phantom, "\in"] & z^{\mu} \xi^{-1}(\tilde{u}_-) \arrow[u, phantom, "\in"] \end{tikzcd}$$

it is inverse to $R^{\lambda}_{\mu} \to T^{\lambda}_{\mu}$

So, $R^{\lambda}_{\mu} \simeq T^{\lambda}_{\mu} = U_- \cdot z^{\mu}$ ← affine space

Claim 2 $\underset{\text{loop rotation}}{\mathbb{C}^{\times} \curvearrowright} W^{\lambda}_{\mu}$ contracts it to R^{λ}_{μ}

(in particular $(W^{\lambda}_{\mu})^{\mathbb{C}^{\times}} = R^{\lambda}_{\mu}$)

prf. $R^{\lambda}_{\mu} \leftarrow \mathbb{C}^{\times}$- fixed by Claim 1 (use that $\mathbb{C}^{\times} \curvearrowright Gr^{\lambda}$ is trivial)

$t.(u h z^{\mu} u_-) = \underset{\downarrow \atop 1}{u(t^{-1}z)} \underset{\downarrow \atop 1}{h(t^{-1}z)} t^{-\mu} z^{\mu} u_-(t^{-1}z) t^{\mu}$

$\exists \lim_{t \to \infty} t^{-\mu} u_-(t^{-1}z) t^{\mu} \in U_-[[z^{-1}]]_1$

$a_1 z^{-1} + a_2 z^{-2} + \ldots$
$\downarrow$
$a_1 t z^{-1} + a_2 t^2 z^{-2} + \ldots$
$\lim_{t\to\infty} \downarrow \cdot t^{\langle -\mu, -\alpha \rangle} = t^{\langle \mu, \alpha \rangle}$
0 or $a_1 z^{-1}$

$\alpha \in \Delta_+$
$\langle \mu, \alpha \rangle \geq -1$

see that the limit is indeed in R^{λ}_{μ}.

Cor. $W^{\lambda}_{\mu} \simeq \mathbb{A}^{\langle 2\rho, \lambda - \mu \rangle}$

prf. $\exists \mathbb{C}^{\times} \curvearrowright W^{\lambda}_{\mu} \xrightarrow{\text{contracts}} R^{\lambda}_{\mu} \simeq \mathbb{A}^{\langle \rho, \lambda - \mu \rangle}$

$R^\lambda_\mu \; \sigma$ repellent to $z^\mu \in W^\lambda_\mu$ under (2ρ)

Take $\mathbb{C}^\times \to \mathbb{C}^\times \times T$; $t \mapsto (t^d, (-2\rho)(t))$, $d \gg 0$

Then $\mathbb{C}^\times \curvearrowright W^\lambda_\mu$ contracts to $\{z^\mu\}$, W^λ_μ - smooth

$$\Big\Downarrow \; BB$$

$$W^\lambda_\mu \simeq T^* \mathbb{A}^{\langle 2\rho, \lambda - \mu\rangle}$$

$$\nearrow$$

- can construct explicit identification (i.e. $W^\lambda_\mu \simeq T^* \mathbb{A}^{\langle \rho, \lambda-\mu\rangle}$)
- can show that symplectic structure on W^λ_μ is standard
- can quantize to explicit identification $Y^\lambda_\mu \simeq \mathcal{D}(\mathbb{A}^{\langle \rho, \lambda-\mu\rangle})$

$$\begin{aligned} &\cup\!\!\!\cup \\ &E_i^{(1)} \mapsto x_i \\ &F_i^{(1)} \mapsto \frac{\partial}{\partial x_i} \end{aligned}$$

One concrete corollary of the Thm:

$$\mathrm{ch}_T \, T_{z^{\underline{\mu}}} \widetilde{W}^{\underline{\lambda}}_{\underline{\mu}} = \sum_{i=1}^{m} \sum_{\substack{\alpha^\vee \in \Delta^\vee_- \\ \langle \alpha^\vee, \mu_i\rangle = 1}} (t^{\alpha^\vee} + t^{-\alpha^\vee})$$

$$\mathrm{ch}_{T \times \mathbb{C}^\times} T_{z^{\underline{\mu}}} \widetilde{W}^{\underline{\lambda}}_{\underline{\mu}} = \sum_{i=1}^{m} \sum_{\substack{\alpha^\vee \in \Delta^\vee_- \\ \langle \alpha^\vee, \mu_i\rangle > 0}} \left(\hbar^{-\langle \alpha^\vee, \mu_1 + \ldots + \mu_{i-1}\rangle} t^{\alpha^\vee} + \hbar^{1 + \langle \alpha^\vee, \mu_1 + \ldots + \mu_{i-1}\rangle} t^{-\alpha^\vee} \right)$$

$$\Delta^\vee_{\mu_i,-} = \{\alpha^\vee \in \Delta^\vee_- ; \langle \alpha^\vee, \mu_i\rangle = 1\}$$

$\Downarrow$ use BB decomp.

$$\sum_{n \in \mathbb{Z}_{\geq 0}} \dim H^n_c (\widetilde{W}^{\underline{\lambda}}_{\underline{\mu}}) q^n = \sum_{\substack{\underline{\mu}, |\underline{\mu}| = \mu \\ \mu_i \in W\lambda_i}} q^{2\left(\sum_{i=1}^n |\Delta^\vee_{\mu_i,-}| + \sum_{i=1}^n \left|\{\alpha^\vee \in \Delta^\vee_{\mu_i,-}, \langle \alpha^\vee, \mu_1 + \ldots + \mu_{i-1}\rangle = -1\}\right|\right)}$$

$\nwarrow$

Can compute

Poincare pol. of $\widetilde{W}^{\underline{\lambda}}_{\underline{\mu}}$