Lecture 21

суббота, 19 апреля 2025 г. 13:04

Last time:

$\overline{\mathcal{W}}^{\lambda}_{\mu}$ ⟜ smooth if λ ⟜ minuscule

(use that $\langle \lambda, \alpha \rangle \in \{-1,0,1\}\ \forall \alpha \in \Delta$)

$\overline{\mathrm{Gr}_G^{\lambda}} = \mathrm{Gr}^{\lambda}_G = G/P_{\lambda}$ ⟜ λ - minuscule

Type A ⟜ any fundamental is minuscule

Prop: $\overline{\mathcal{W}}^{\lambda}_{\mu}$ has a resolution if

$$\lambda = \lambda_1 + \dots + \lambda_m$$

minuscule

Rmk: it is not iff ↺ (for example $\overline{\mathcal{W}}^{\lambda}_{\lambda} = \mathcal{W}^{\lambda}_{\lambda}$ always has a resolution)

Construction of the resolution:

def. makes sense for arbitrary λ_i, will be nonsing if λ_i ⟜ minuscule

$$\widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu} := \left\{ \begin{array}{l} (\mathcal{P}^{\mathrm{triv}} = \mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_m, \sigma_1, \dots, \sigma_m, \mathcal{P}^B) \\ \mathcal{P}_i \text{ ⟜ } G\text{-bundle on } \mathbb{P}^1 \\ \sigma_i : \mathcal{P}_{i-1}|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{P}_i|_{\mathbb{P}^1 \setminus \{0\}} \text{ ⟜ } \underline{\text{iso}} \text{ that has pole of deg} \leq \lambda_i \text{ at } 0 \\ \mathcal{P}^B \subset \mathcal{P}_m \text{ ⟜ } \deg \mathcal{P}^B = w_0\mu;\ \mathcal{P}^B|_{\infty} \underset{\text{w.r.t. } \sigma_m \circ \sigma_{m-1} \circ \dots \circ \sigma_1}{=} w_0 B \end{array} \right.$$

Claim: $\widetilde{W}^{\underline{\lambda}}_{\mu} \xrightarrow{\pi} \overline{W}^{\lambda}_{\mu}$ ← symplectic resolution of sing

$$\begin{tikzcd}
\widetilde{W}^{\underline{\lambda}}_{\mu} \arrow[r, "\pi"] \arrow[d, phantom, "\ni" rotate=90] & \overline{W}^{\lambda}_{\mu} \arrow[d, phantom, "\ni" rotate=90] \\
(\mathcal{P}_0, \dots, \mathcal{P}_m, \mathcal{P}^B) \arrow[r, mapsto] & (\mathcal{P}_0, \mathcal{P}_m, \mathcal{P}^B)
\end{tikzcd}$$

Rmks:

$$\begin{tikzcd}
\widetilde{W}^{\underline{\lambda}}_{\mu} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Gr}^{\underline{\lambda}}_{G} \\
\overline{W}^{\lambda}_{\mu} \arrow[r] & \overline{Gr}^{\lambda}_{G} \arrow[from=1-2, to=2-2, crossing over]
\end{tikzcd}$$

← same def. as above but without B-structure

this diagram is cartesian, so to prove that π is a resolution of sing. it is enough to check that

$\widetilde{Gr}^{\underline{\lambda}}_{G} \to \overline{Gr}^{\lambda}_{G}$ ← resol. of sing

sometimes denoted by $Gr^{\lambda_1} * \dots * Gr^{\lambda_m}$

$$\| \quad \color{red}{(*)}$$

$$\left\{ [(g_1, \dots, g_m)] \in G(K) \times_{G(\mathcal{O})} \times \dots \times Gr_G \;\middle|\; \begin{array}{l} G(\mathcal{O})\, g_i\, G(\mathcal{O}) = G(\mathcal{O})\, z^{\lambda_i} G(\mathcal{O}) \\ \text{i.e. pole of } g_i \text{ at } 0 \text{ has deg } \lambda \end{array} \right\}$$

Ex. $\widetilde{Gr}^{\underline{\lambda}}_{G}$ ⊸ iterated byndle with fibers $Gr^{\lambda_i}_{G}$

$\Downarrow$

is smooth

Ex. p ⟜ iso over $p^{-1}(\mathrm{Gr}_G^{\lambda})$

(very similar to Bott-Samelson resolutions of Schubert var in $\mathcal{B}$)

Geometry of $\widetilde{W}^{\underline\lambda}_\mu$

Note $T \curvearrowright \widetilde{W}^{\underline\lambda}_\mu, \overline{W}^{\lambda}_\mu$ ⟜ changing triv σ

Claim: $(\widetilde{W}^{\underline\lambda}_\mu)^T \leftrightarrow \left\{ \underline\mu = (\mu_1, \dots, \mu_m) \;\middle|\; \begin{array}{l} \mu_i \text{ ⟜ weight of } V_{\lambda_i} \\ \Uparrow \\ \mu_i \in W\lambda_i \end{array} \right\}$

the corresp. point will be denoted $z^{\underline\mu}$

prf. $\widetilde{W}^{\underline\lambda}_\mu = \widetilde{\mathrm{Gr}}^{\underline\lambda} \times_{\overline{\mathrm{Gr}}^{\lambda}_G} \overline{W}^{\lambda}_\mu$

$\Downarrow$

$\overline{W}^{\underline\lambda}_\mu = \left\{ (p_1, p_2) \in (\widetilde{\mathrm{Gr}}^{\underline\lambda})^T \times (\overline{W}^{\lambda}_\mu)^T \;\middle|\; \begin{array}{l} \text{images of } p_1, p_2 \\ \text{in } \mathrm{Gr}^{\lambda} \text{ are equal} \end{array} \right\}$

① $(\widetilde{\mathrm{Gr}}^{\underline\lambda})^T \leftrightarrow \left\{ \underline\mu = (\mu_1, \dots, \mu_m) \;\middle|\; \mu_i \in W\lambda_i \right\}$

② $\overline{W}^{\lambda}_{\mu} = \left\{ \begin{matrix} z^{\mu} & \text{if } \mu \text{ a weight of } V_{\lambda} \\ \emptyset & \text{otherwise} \end{matrix} \right\}$

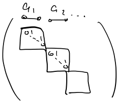
① + ② imply the claim

How ① works for $G = GL_n$:

Recall $Gr_{GL_n} = \{ L \subset \mathbb{C}^n((z)) \mid L \text{ a lattice} \}$

Take $\lambda = \omega_{k_1} + \dots + \omega_{k_m} = (a_1, \dots, a_n)$

Then: $Gr^{\lambda}_{GL_n} = \Big\{ L \subset L_0 \;\Big|\;$ the JNF of $z \curvearrowright L_0/L$ is



prf. $L = g \cdot z^{\lambda}(L_0)$, where $g \in GL_n[[z]]$

Now, $L_0/L = gL_0 / g(z^{\lambda}L_0) = g\left(L_0 / z^{\lambda}L_0\right)$

$\Downarrow$

type of $z \cdot \curvearrowright L_0/L$ is the same as

for $L = z^{\lambda}L_0$.

Clearly: $z^{\lambda} L_0 = L_\lambda = \mathrm{Span}_{\mathbb{C}[[z]]}(z^{a_1}e_1, \dots, z^{a_n}e_n)$

We see that $L_0/L_\lambda = \underset{\substack{\circlearrowleft \\ \cdot z}}{\mathbb{C}[z]/(z^{a_1})} \oplus \dots \oplus \underset{\substack{\circlearrowleft \\ \cdot z}}{\mathbb{C}[z]/(z^{a_n})}$

In the opposite direction: fix $L \subset L_0$, we know

$\exists \underset{\substack{\shortparallel \\ (a_1', \dots, a_n')}}{\lambda'} \in \Lambda^+$ s.t. $L = \underset{!!}{g \cdot z^{\lambda'}(L_0)} \subset L_0$

$$z^{\lambda'} L_0 \subset L_0$$

$\underline{a_i' \geq 0} \ \forall i \Leftarrow$

It remains to note that $z \cdot \circlearrowright L_0/z^{\lambda'}L_0$ with

JNF $(a_1', \dots, a_n')$

So, we must have $\lambda = \lambda'$.

Cor: $\overline{Gr_G^{\lambda}} = \left\{ L \subset L_0 \;\middle|\; \begin{array}{l} \text{type of } \cdot z \\ \text{on } L_0/L \text{ is } \leq \lambda \end{array} \right\}$

Now, $\lambda = \overset{\substack{\lambda_1 \\ \shortparallel}}{\omega_{k_1}} + \dots + \overset{\substack{\lambda_m \\ \shortparallel}}{\omega_{k_m}}$, then similarly to above use description (*)

one can show that:

$$\begin{tikzcd} \widetilde{Gr}^{\lambda}_{GL_n} \arrow[d] & = \left\{ L_{-m} \overset{k_m}{\subset} L_{-m+1} \overset{k_{m-1}}{\subset} \dots \subset L_{-1} \overset{k_1}{\subset} L_0 \;\middle|\; \begin{array}{l} \dim L_{1-i}/L_{-i} = k_k \\ z(L_{-i}) \subset L_{-i-1} \end{array} \right\} \arrow[d, mapsto] \\ \widetilde{Gr}^{\lambda}_{GL_n} & \ni L_{-m} \subset L_0 \end{tikzcd}$$

$\cdot z$ $\quad$ $\cdot z$

iterated bundle

of $Gr^{\omega_i} \simeq G/P_{\omega_i} = Gr(i,n)$ $\quad$ mod out by $z^{-1}L_0$

$$\left\{ L \subset L_0 \;\middle|\; \begin{array}{l} \dim L_0/L = i \\ z(L_0) \subset L \end{array} \right\} \Longleftrightarrow \left\{ z^{-1}L_0 \subset L \subset L_0 \right\}$$

We see that $\widetilde{Gr}^{\lambda}_{GL_n} \hookrightarrow \overbrace{Gr_{GL_n} \times \dots \times Gr_{GL_n}}^{m}$

$$(L_\bullet \subset L_0) \longmapsto (L_{-1}, \dots, L_{-m})$$

So, every T-fixed point on $\widetilde{Gr}^{\lambda}$ is of the

form: $L_{\upsilon_m} \subset L_{\upsilon_{m-1}} \subset \dots \subset L_{\upsilon_1} \subset L_0$ for

some cocharacters υ_i (recall $L_{\upsilon_i} = z^{\upsilon_i} \cdot L_0$).

Remains to note $\mu_i := \upsilon_i - \upsilon_{i-1} \in W \cdot \omega_{k_i}$

$$\Uparrow$$

$$\left(L_{\mu_i} \in Gr^{\omega_{k_i}} = G/P_{\omega_{k_i}} \right)$$

T - fixed point

$\{z^{\mu}\}$

Now, $\left(\overline{W}^{\lambda}_{\mu}\right)^T = \left(U[[z^{-1}]]_1 \, T[[z^{-1}]]_1 \, z^{\mu} \, U_-[[z^{-1}]]_1 \right)^T \cap$

$$\cap \; \overline{G[z] \, z^{\lambda} \, G[z]}$$

contains z^{μ} iff μ ← weight of V_λ

Upshot:

$$\begin{tikzcd} \widetilde{\mathcal{W}}^{\lambda}_{\mu} \arrow[d, "\text{resolution}"'] \\ \overline{\mathcal{W}}^{\lambda}_{\mu} \end{tikzcd}$$

T-fixed points are isolated $\rightsquigarrow$ $\mu = \mu_1 + \dots + \mu_m$
$\mu_i \in W\lambda_i$

in particular, can study cat $\mathcal{O}$.

Local geometry of $\widetilde{\mathcal{W}}^{\lambda}_{\mu}$ and construction of Verma-type modules

Recall $\rightsquigarrow$ Verma modules generate cat. $\mathcal{O}$.
are in bijection with T-fixed points
(for generic quant. param)

Goal: see this bijection from geometry and construct Verma modules explicitly (will later help to prove quant. Hikita-Nakajima)

Symmetric def. of $\overline{\mathcal{W}}^{\lambda}_{\mu}$:

Fix $\mu = \mu_- + \mu_+$.

Prop: $\overline{\mathcal{W}}^{\lambda}_{\mu} \simeq \overline{\mathcal{W}}^{\lambda}_{\mu_-,\mu_+} := \Big\{ (\mathcal{P}_0^{B_-}, \mathcal{P}_1, \dots, \mathcal{P}_m, \mathcal{P}_m^{B})$

$\sigma_i : \mathcal{P}_i|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{P}_{i+1}|_{\mathbb{P}^1 \setminus \{0\}}$
$\deg \leq \lambda_i$

$s = s_0 \rightsquigarrow$ triv. of $\mathcal{P}_0$ at ∞

$$\left|\begin{array}{l} \deg \mathcal{T}_0^{B_-} = -w_0\mu_- \, ; \quad \deg \mathcal{T}_m^{B} = w_0\mu_+ \\ \mathcal{T}_0^{B_-}\big|_\infty = B \, ; \quad \mathcal{T}_m^{B}\big|_\infty = B_- \end{array}\right.$$

prf. Do Hecke transformation at ∞ and twist all of the bundles by $w_0\mu_-$

arbitrary collections of dom. weights

Cor: for any $\underline{\lambda}, \underline{\lambda}'$; $\mu \le \lambda$, $\mu' \le \lambda'$

there exists an open embedding:

$$\widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu} \times \mathcal{W}^{\underline{\lambda}'}_{\mu'} \hookrightarrow \widetilde{\mathcal{W}}^{(\underline{\lambda},\underline{\lambda}')}_{\mu+\mu'}$$

prf:

$$\widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu} \simeq \widetilde{\mathcal{W}}^{\underline{\lambda}}_{\mu,0} \, ; \quad \mathcal{W}^{\underline{\lambda}'}_{\mu'} \simeq \widetilde{\mathcal{W}}^{\underline{\lambda}'}_{0,\mu'} \, ; \quad \widetilde{\mathcal{W}}^{(\underline{\lambda},\underline{\lambda}')}_{\mu+\mu'} \simeq \widetilde{\mathcal{W}}^{(\underline{\lambda},\underline{\lambda}')}_{\mu,\mu'}$$

$$\begin{gathered} \rotatebox{90}{$\in$} \qquad\qquad\qquad\qquad \rotatebox{90}{$\in$} \\ \left(\mathcal{T}_0^{B_-}, \mathcal{T}_1, \dots, \mathcal{T}_m \simeq \mathcal{T}^{triv}\right) \times \left(\mathcal{T}^{triv} \simeq \mathcal{T}_0', \mathcal{T}_1', \dots, \mathcal{T}_{m'}', \mathcal{T}_{m'}'^{,B}\right) \\ \big\downarrow \\ \left(\mathcal{T}_0^{B_-}, \mathcal{T}_1, \dots, \mathcal{T}_{m-1}, \mathcal{T}^{triv}, \mathcal{T}_1', \dots, \mathcal{T}_{m'}', \mathcal{T}_{m'}'^{,B_-}\right) \\ \cap \\ \widetilde{\mathcal{W}}^{(\underline{\lambda},\underline{\lambda}')}_{\mu,\mu'} \end{gathered}$$

point here lies in the image iff the m'th bundle is trivial, this is open condition!

Passing to slices get map:

$$\overline{W}^{\lambda}_{\mu} \times \overline{W}^{\lambda'}_{\mu'} \xrightarrow[\underline{\text{birational}}]{} \overline{W}^{\lambda+\lambda'}_{\mu+\mu'}$$

↖ will be quantized to comult. morphism

Ex:

$$\overline{W}^{(N,0)}_{(N-k,k)} = \left\{ M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \middle| \begin{array}{l} x_{22} = z^{k} + \dots \\ \deg x_{21}, x_{12} < k \\ \det M = z^{N} \end{array} \right\}$$

$$\begin{tikzcd}
\overline{W}^{(N,0)}_{(N-k,k)} \arrow[r, phantom, "\times"] & \overline{W}^{(N',0)}_{(N'-k',k')} \arrow[r] & \overline{W}^{(N+N',0)}_{(N+N'-k-k',\, k+k')} \\
\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \arrow[u, phantom, "\cup"] \arrow[r, phantom, "\times"] & \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & \begin{pmatrix} A' & B' \\ C' & D \end{pmatrix}
\end{tikzcd}$$

$B' \longrightarrow B$

divide with a remainder by D

$C' \longrightarrow C$

$$A = \frac{z^{k_1+k_2} + BC}{D}$$

Then, $m(M_1, M_2) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$

(Ex)

Ex: if $k \geq N$ then multiplication morphism is exactly the product ↖ note that μ is anti dominant in this case.

In general: product in matrix terms is described as follows.

$$\mathbb{W}_\mu = U[[z^{-1}]]_1 \, T[[z^{-1}]]_1 \, z^\mu \, U_-[[z^{-1}]]_1$$

$$\wr$$

$$U[z^{-1}] \backslash U((z^{-1})) \, T[[z^{-1}]]_1 \, z^\mu \, U_-((z^{-1})) / U[z^{-1}]$$

use: $U[z^{-1}] \times U[[z^{-1}]]_1 \xrightarrow{\sim} U((z^{-1}))$

(check for $U = \mathbb{G}_a$, deduce in general)

$g \in \mathbb{W}_\mu, \quad g' \in \mathbb{W}_{\mu'}$

$\Downarrow$

$$g \cdot g' \in U[[z^{-1}]]_1 \, T[[z^{-1}]]_1 \, z^\mu \underbrace{U_-[[z^{-1}]]_1 \, U[[z^{-1}]]_1}_{\begin{array}{c}\cap \\ U[[z^{-1}]]_1 \, T[[z^{-1}]]_1 \, U_-[[z^{-1}]]_1\end{array}} T[[z^{-1}]]_1 \, z^{\mu'} \, U_-[[z^{-1}]]_1$$

$\Downarrow$

$$g g' \in U((z^{-1})) \, T[[z^{-1}]] \, z^{\mu+\mu'} \, U_-((z^{-1}))$$

$\exists! \; u \in U[z], \; u_- \in U_-[z]$ s.t.

$$u g g' u_- \in \mathbb{W}_{\mu+\mu'}$$

$$\begin{array}{ccc} \mathbb{W}_\mu \times \mathbb{W}_{\mu'} & \to & \mathbb{W}_{\mu+\mu'} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ (g, g') & \mapsto & u g g' u_- \end{array}$$

(Ex)

Rem: if μ, μ' ⊸ anti-dominant

$\Downarrow$

$$\underline{m(gg') = gg'}$$