Lecture 20

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Last time:

$$\overline{\mathcal{W}}^{\lambda}_{\mu} = U[[z^{-1}]]_1 \, T[[z^{-1}]]_1 \, z^{\mu} \, U_-[[z^{-1}]]_1 \cap \overline{G[z] z^{\lambda} G[z]}$$

defined for any $\mu \leq \lambda$

$G = GL_2$, $\lambda = (N, 0) = N\omega_1$, $\mu = (N-k, k) = N\omega_1 - k\alpha$

$$\overline{\mathcal{W}}^{(N,0)}_{(N-k,k)} = \left\{ M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \;\middle|\; \begin{array}{l} x_{ij} \in \mathbb{C}[z] \\ x_{22} = z^k + \text{lower} \\ \deg x_{12}, x_{21} < k \\ \det M = z^N \end{array} \right\}$$

Ex: $N = 0 \Rightarrow \mu = (-k, k) \leftarrow$ antidominant

See that
$$\overline{\mathcal{W}}^{0}_{(-k,k)} = \left\{ (x_{12}, x_{22}) \;\middle|\; \begin{array}{l} x_{22} = z^k + \text{lower} \\ \deg x_{12} < k \\ \gcd(x_{12}, x_{22}) = 1 \end{array} \right\}$$

$$\Big\|$$

$$\left\{ \begin{array}{c} f : \mathbb{P}^1 \to \mathbb{P}^1 \\ \rotatebox{90}{$\in$} \qquad \rotatebox{90}{$\in$} \\ z \mapsto \dfrac{x_{22}(z)}{x_{12}(z)} \end{array} \;\middle|\; \begin{array}{l} f(\infty) = \infty \\ \deg f = k \end{array} \right\}$$

$[1 : z] \mapsto [x_{12}(z) : x_{22}(z)]$

$\|$

$[z^{-1} : 1] \mapsto [z^{-k} x_{12}(z) : z^{-k} x_{22}(z)]$

$\|$

$[z^{-1}? : 1 + z^{-1}?]$

$[0 : 1] \mapsto [0 : 1]$

$\| \qquad \|$

$\infty \qquad \infty$

$f^*(\mathcal{O}(-1) \hookrightarrow \mathcal{O}^{\oplus 2})$

Note: $f : \mathbb{P}^1 \to \mathbb{P}^1 \iff \mathcal{L} \hookrightarrow \mathcal{O}^{\oplus 2}$

$\deg f = \deg \mathcal{L}$

So:
$$\overline{\mathcal{W}}^{0}_{(-k,k)} = \left\{ \mathcal{O}(-k) \overset{2}{\hookrightarrow} \mathcal{O}^{\oplus 2} \;\middle|\; 2|_{\infty} = \left(\begin{array}{ccc} \mathbb{C} & \hookrightarrow & \mathbb{C} \oplus \mathbb{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ 1 & \longmapsto & (0,1) \end{array} \right) \right\}$$

embedding of vector bundles

$$\Big\|$$

$$\left\{ \begin{array}{l} B\text{-structure} \quad \mathcal{P}^B \subset \mathcal{P}^{\text{triv}} \leftarrow \text{trivial } GL_2\text{-bundle on } \mathbb{P}^1 \\ \deg \mathcal{P}^B = (k,-k) \;;\; \mathcal{P}^B|_\infty = B_- \end{array} \right\}$$

Recall: $\mathcal{P} \leftarrow G$-bundle on $\mathbb{P}^1$ $\Rightarrow$ B-structure in $\mathcal{P}$ is a B-subtorsor $\mathcal{P}^B \subset \mathcal{P}$.

$\deg \mathcal{P}^B = \deg\left(\mathcal{P}^B/U\right) \leftarrow T$-bundle that is isomorphic to $\mathcal{O}(n_1) \oplus \dots \oplus \mathcal{O}(n_r)$ $(r = \dim T)$

$\deg \mathcal{P}^B = (n_1, \dots, n_r)$.

Claim:

$$\overline{W}^{\lambda}_{\mu} = \left\{ \begin{array}{c} (\mathcal{P}, \sigma, \mathcal{P}^B) \\ \mathcal{P}^B \subset \mathcal{P} \\ \uparrow \\ B\text{-subtorsor} \end{array} \middle| \begin{array}{l} (\mathcal{P}, \sigma) \in Gr_G \\ \deg \mathcal{P}^B = w_0 \mu \\ \mathcal{P}^B|_\infty = B_- \end{array} \right\}$$

prf. pick $(\mathcal{P}, \sigma, \mathcal{P}^B)$, let's construct an element of $\overline{W}^{\lambda}_{\mu}$.

Let $\mathcal{P}^{B_-} \leftarrow B_-$-bundle of deg μ corresp to $\mathcal{P}^B$
(recall $B_- = w_0 B w_0^{-1}$)

Consider $\mathcal{P}|_{D_\infty} \leftarrow$ contains B_--structure $\mathcal{P}^{B_-}|_{D_\infty}$

and B-structure $\underset{\text{standard}}{\mathcal{P}^{B}_{D_\infty}} \subset \mathcal{P}^{\text{triv}}_{D_\infty}$

Ex

They are transversal $\Rightarrow \exists!$ T-structure $\mathcal{P}^{T}|_{D_\infty}$

s.t. $\operatorname{Ind}_T^{B_-} \mathcal{P}^T|_{D_\infty} = \mathcal{P}^{B_-}|_{D_\infty}$, $\operatorname{Ind}_T^{B} \mathcal{P}^T|_{D_\infty} = \mathcal{P}^{B}_{D_\infty}$

(Ex: B-structures : $\mathcal{L} \hookrightarrow \mathcal{E}$ $\leftarrow$ transversal if $\forall x \in \mathbb{P}^1$
$\mathcal{L}' \hookrightarrow$ $\qquad \mathcal{L}_x + \mathcal{L}'_x = \mathcal{E}_x$
$\Leftarrow$
$\mathcal{L}' \simeq \mathcal{E}/\mathcal{L} \Rightarrow \mathcal{E} \simeq \mathcal{L} \oplus \mathcal{L}' \leftarrow$ splitting gives T-structure)

Consider ${}'\mathcal{P}^T := \mathcal{P}^T(-\mu \cdot \infty)$ (think about the corresp. line bundles)

Note: ${}'\mathcal{P}^T$, $\mathcal{P}^T \leftarrow$ are equal on $\mathring{D}_\infty$

Define ${}'\mathcal{P}^{B_-} \leftarrow$ gluing of $\mathcal{P}^{B_-}$ and ${}'\mathcal{P}^T \times^T B_-$ along $\mathring{D}_\infty$.

$\leftarrow$ this is called "Hecke transform" at ∞

Ex: $\deg({}'\mathcal{P}^{B_-}) = 0 \Rightarrow$ it reduces to U_--bundle ${}'\mathcal{P}^{U_-}$. The induced G-bundle ${}'\mathcal{P}$ must be trivial as well as ${}'\mathcal{P}^{U_-}$ (Ex, look at $G = GL_n$)

So, we have trivializations:

$$\begin{tikzcd}
{}'\mathcal{P} & \mathcal{P}^{\text{triv}} \arrow[l, "\simeq"'] & (*) \\
{}'\mathcal{P}^{B_-} \arrow[u, phantom, "\cup"] & \mathcal{P}^{\text{triv},B_-} \arrow[l, "\simeq"'] \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

$({}'\mathcal{P}^{B,-}$ and $\mathcal{P}^{B_-})$

Recall now that ${}'\mathcal{P}$ and $\mathcal{P}$ are equal away from ∞ so $(*)$ induces:

$$\sigma_{\mathbb{A}^1} : \mathcal{P}^{\text{triv}}_{\mathbb{A}^1} \xrightarrow{\simeq} \mathcal{P}|_{\mathbb{A}^1}$$

Composing $\sigma^{-1} \circ \sigma_{\mathbb{A}^1}$ we obtain the element of $\overline{G[z]\, z^{\lambda}\, G[z]}$

$\nearrow$

(use that no poles at 0)

Consider the embedding $G[z^{\pm 1}] \hookrightarrow G((z^{-1}))$,

this corresponds to restricting $\sigma^{-1} \circ \sigma_{/A'}$ to $\mathring{D}_\infty$

The resulting element g will lie in:

$$U[[z^{-1}]]_1 \cdot T[[z^{-1}]]_1 z^{\mu} \cdot U_-[[z^{-1}]]_1$$

(for example if $\mu = 0$, then we know that $\sigma_{/A'}$ extends to ∞ $\Rightarrow$ we get an element of $G[[z^{-1}]]_1$
$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \parallel$
$U[[z^{-1}]]\, T[[z^{-1}]]_1\, U_-[[z^{-1}]]_1$)

in general use that $\sigma_{/A'}$ extends to ∞ after Hecke transform as above.

We see that $g \in \overline{\mathcal{W}}_{\mu}^{\lambda}$.

Another approach (without Hecke transformations)

Fix trivializations: $\sigma_{B_-} : \mathcal{P}^{\mathrm{triv}, B_-}\big|_{/A'} \simeq \mathcal{P}^{B_-}\big|_{/A'}$

that is equal to id on (σ_{B_-}/U)
(recall that $\mathcal{P}^{B_-}/U \simeq \mathcal{O}(z^{\mu})$)

$$\sigma_u : \mathcal{P}^{\mathrm{triv}, U}\big|_{/A'} \simeq \mathcal{P}^{\mathrm{triv}, U}\big|_{/A'}$$

Then $\sigma : \mathcal{P}^{\mathrm{triv}}\big|_{\mathbb{P}^1 \setminus \{0\}} \simeq \mathcal{P}\big|_{\mathbb{P}^1 \setminus \{0\}}$ defines an

element of $\overline{G[z]\, z^{\lambda}\, G[z]} \subset G(z)$ given by

$$\mathrm{Ind}_u^G(\sigma_u^{-1}) \circ \sigma^{-1} \circ \mathrm{Ind}_{B_-}^G(\sigma_{B_-}).$$

note that they have no pole at 0

This element is well-defined up to the right multiplication by $U_-[z]$ and left multiplication by $U[z]$.

Consider the embedding $G[z^{\pm 1}] \hookrightarrow G((z^{-1}))$,

(i.e. restrict to $\mathring{D}_\infty$)

using that $\deg \mathcal{P}^{B_-} = \mu$ and $\mathcal{P}^{B_-}|_\infty = B_-$

we obtain an element of:

$$U[z] \backslash U((z^{-1}))\, z^\mu\, T[[z^{-1}]]_1 U_-((z^{-1})) / U_-[z]$$

element here lifts uniquely to an element

$$g \in U[[z^{-1}]]_1\, z^\mu\, T[[z^{-1}]]_1\, U_-[[z^{-1}]]_1$$

$$\left(\text{use that } \underset{\substack{\text{check for } \mathbb{G}_a\text{, deduce}\\ \text{in general}}}{U[[z^{-1}]]_1 \times U[z] \xrightarrow{\sim}} U((z^{-1})) \Rightarrow \begin{array}{c} U((z^{-1}))/U[z] \\ \uparrow\wr \\ U[[z^{-1}]]_1 \end{array} \right)$$

So g is the desired element of $\overline{W}^\lambda_\mu$ ∎

Let's construct the inverse map.

Pick $g \in \overline{W}^\lambda_\mu \subset G[z^{\pm 1}]$, it defines a G-bundle $\mathcal{P}$ together with trivializations

$\sigma_0 : \mathcal{P}^{triv}_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1 \setminus \{0\}}$

$\sigma_\infty : \mathcal{P}^{triv}_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{A}^1}$

Let $\mathcal{G}^{B_-}_{/\mathbb{A}^1}$ be the image of the standard B_--structure in $\mathcal{G}^{triv}_{/\mathbb{A}^1}$. It extends to the degree μ B_--structure $\mathcal{G}^{B_-}$ in $\mathcal{G}$ by setting $\mathcal{G}^{B_-}_\infty := B_-$.

Then the inverse map sends $g \mapsto (\mathcal{G}, \sigma_0, \mathcal{G}^B)$

B-bundle corresp. to $\mathcal{G}^{B_-}$

Ex: check that two maps above are indeed inverse.

Example: $\lambda = 0 \Rightarrow (\mathcal{G}, \sigma) = (\mathcal{G}^{triv}, \sigma_{triv})$, so

$$\overline{W}^\circ_\mu = \left\{ \mathcal{G}^B \subset \mathcal{G}^{triv} \;\middle|\; \begin{array}{l} \deg \mathcal{G}^B = w_0\mu \\ \mathcal{G}^B|_\infty = B_- \end{array} \right\}$$

$$\|$$

$$\left\{ f : \mathbb{P}^1 \to \mathcal{B} \;\middle|\; \begin{array}{l} \deg f = w_0\mu \\ f(\infty) = B_- \end{array} \right\} = \overset{\circ}{Z}{}^{-w_0\mu}$$

flag variety

$$\mathcal{G}^B = f^*\begin{pmatrix} G \\ \downarrow \\ G/B \end{pmatrix}$$

open Zastava spaces

Resolutions of $\overline{W}^\lambda_\mu$

Using moduli space definition of $\overline{W}^\lambda_\mu$ can construct their resolutions.

(then we do not need to resolve)

Question: when $\overline{W}^\lambda_\mu$ ← smooth?

Answer: smooth locus of $\overline{W}^\lambda_\mu$ is:

$$W^\lambda_\mu = \overline{W}^\lambda_\mu \cap G[z]\, z^\lambda\, G[z]$$

So, $\overline{W}^\lambda_\mu$ is smooth if $W^\lambda_\mu = \overline{W}^\lambda_\mu$

that, in particular, holds if:

$$\overline{G[z]\, z^\lambda\, G[z]} = G[z]\, z^\lambda\, G[z]$$

$$\overline{G[z]\, z^\lambda\, G[z]} = \bigsqcup_{\substack{\mu \le \lambda \\ \mu \in \Lambda^+}} G[z]\, z^\mu\, G[z]$$

this is <=> to:

there are **no** dominant $\mu \in \Lambda^+$ s.t. $\mu \le \lambda$

equivalently, all weights of V_λ are in $W\cdot\lambda$ ← such λ are called **minuscule**

(use that if μ is a weight of V_λ => μ^+ ← dominant representative of μ is also a weight of V_λ)

Ex: $\lambda = 0$ ← always works

(in particular $\overline{W}^0_\mu$ ← is always smooth)

$\mathfrak{g} = \mathfrak{gl}_n$ => $\lambda = \omega_i = (\underbrace{1, \dots, 1}_{i}, 0, \dots, 0)$

$V_\lambda = \Lambda^i \mathbb{C}^n \leftarrow$ weight spaces are $e_{k_1} \wedge \dots \wedge e_{k_i}$

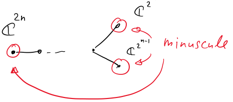
all in the same S_n-orbit

In general : $\lambda \leftarrow$ minuscule $\Rightarrow$ λ-fundamental

$\cancel{\Leftarrow}$

(for example in type $E_8 \leftarrow$ NO minuscule)
except $\lambda = 0$

$D_n \leftarrow$



Again see that type A is "better" than other in the sense that in type A any dominant $\lambda \in \Lambda^+$ is the sum of minuscule.

So, $\overline{\mathcal{W}}^\lambda_\mu \leftarrow$ is smooth for $\lambda \leftarrow$ minuscule but not in general.

Resolution of $\overline{\mathcal{W}}^\lambda_\mu$ exists if:

$$\exists\, \lambda = \underbrace{\lambda_1 + \dots + \lambda_m}_{\text{minuscule}} \; ; \; \underline{\lambda} = (\lambda_1, \dots, \lambda_m)$$

def. makes sense for arbitrary λ_i, will be nonsing if $\lambda_i \leftarrow$ minuscule

Then:

$$\widetilde{\mathcal{W}}^{\underline{\lambda}}_\mu = \left\{ \begin{array}{l} (\mathcal{F}^{\text{triv}} = \mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_m, \sigma_1, \dots, \sigma_m, \mathcal{F}^B) \\ \mathcal{F}_i \leftarrow G\text{-bundle on } \mathbb{P}^1 \\ \sigma_i : \mathcal{F}_{i-1}|_{\mathbb{P}^1 \setminus \{0\}} \xrightarrow{\sim} \mathcal{F}_i|_{\mathbb{P}^1 \setminus \{0\}} \leftarrow \text{iso that has pole of deg} \leq \lambda_i \text{ at } 0 \\ \mathcal{F}^B \subset \mathcal{F}_m \leftarrow \deg \mathcal{F}^B = w_0\mu \, ; \; \mathcal{F}^B|_\infty = B_- \text{ w.r.t. } \sigma_m \circ \sigma_{m-1} \circ \dots \circ \sigma_1 \end{array} \right\}$$

convolution diagram over $\overline{\mathcal{W}}^\lambda_\mu$

Claim:

$$\begin{tikzcd} \widetilde{W}^{\lambda}_{\mu} \arrow[r, "\pi"] & \overline{W}^{\lambda}_{\mu} \\ (\mathcal{F}_0, \dots, \mathcal{F}_m, \mathcal{F}^B) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & (\mathcal{F}_0, \mathcal{F}_m, \mathcal{F}_B) \arrow[u, phantom, sloped, "\in"] \end{tikzcd}$$

← symplectic resolution of sing.

Rmks:

$$\begin{tikzcd} \widetilde{W}^{\lambda}_{\mu} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \widetilde{Gr}^{\lambda}_{G} \arrow[d] \\ \overline{W}^{\lambda}_{\mu} \arrow[r] & \overline{Gr}^{\lambda}_{G} \end{tikzcd}$$

← Same def. as above but without B-structure

this is a cartesian diagram, so to prove that

π ← resolution of sing. it is enough to

check that $\widetilde{Gr}^{\lambda}_{G} \xrightarrow{p} \overline{Gr}^{\lambda}_{G}$ ← is a resol. of sing.

sometimes denoted by $Gr^{\lambda_1} * \dots * Gr^{\lambda_m}$

Ex: $\widetilde{Gr}^{\lambda}_{G}$ ← iterated bundle with fibers $Gr^{\lambda_i}_{G}$

$\Downarrow$

is smooth.

Ex: p is iso over $p^{-1}(Gr^{\lambda}_{G})$

(Very similar story to Bott-Samelson resolutions of Schubert var in $\mathcal{B}$)

"Meaning" of $\widetilde{W}^{\lambda}_{\mu}$ ← singular monopoles

on $\mathbb{C}^2 \times \mathbb{R}$, λ_i ← singular charges (colors the puncture)

μ ← total monopole charge

Geometry and representation theory of $\widetilde{W}^{\underline{\lambda}}_{\mu} \to \overline{W}^{\lambda}_{\mu}$

Note: $T \curvearrowright \widetilde{W}^{\lambda}_{\mu}, \overline{W}^{\lambda}_{\mu}$ ← changing trivialization 6

note that description is very explicit and is of "rep. theory" flavor

Claim: $$(\widetilde{W}^{\underline{\lambda}}_{\mu})^T \longleftrightarrow \left\{ \underline{\mu} = (\mu_1, \dots, \mu_n) \;\middle|\; \begin{array}{c} \mu_i \multimap \text{weight of } V_{\lambda_i} \\ \Updownarrow \\ \mu_i \in W \cdot \lambda_i \end{array} \right\}$$

the corresp. point will be denoted by $z^{\underline{\mu}}$

in particular, have isolated fixed points

so can study cat $\mathcal{O}$!

prf: $$\overline{W}^{\underline{\lambda}}_{\mu} \hookrightarrow \widetilde{Gr}^{\underline{\lambda}}_{G} \times_{\overline{Gr}^{\lambda}_{G}} \overline{W}^{\lambda}_{\mu}$$

① $$(\overline{W}^{\lambda}_{\mu})^T = \begin{cases} z^{\mu} & \text{if } \mu \multimap \text{weight of } V_\lambda \\ \emptyset & \text{otherwise} \end{cases}$$

② $$(\widetilde{Gr}^{\underline{\lambda}}_{G})^T \longleftrightarrow \left\{ \underline{\mu} = \mu_1 + \dots + \mu_n \;\middle|\; \mu_1 + \dots + \mu_n \multimap \text{weight of } V_\lambda \right\}$$

① + ② imply the claim

will prove ① and ② next time