Lecture 2

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Nilpotent cone, Springer resolution, Slodowy varieties: basic properties

From last time:

$$X^n_k = \left\{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \middle| \begin{array}{l} D = z^k + \dots \\ \deg B, \deg C < k \\ \det M = z^n \end{array} \right\}$$

$k = 1, \ n \geq 1 \Rightarrow$

$$X^n_1 \simeq \left\{ \begin{pmatrix} \frac{z^n + bc}{z - a} & b \\ c & z - a \end{pmatrix} \middle| \ a^n + bc = 0 \right\}$$

$$\wr\!\wr$$

$$\mathbb{A}^2 / \mathbb{Z}/n\mathbb{Z}$$

Kleinian singularity of type A_{n-1}

$$X^n_1 = \mathbb{A}^2/\mathbb{Z}/n\mathbb{Z} \circlearrowleft \mathbb{C}^\times ; \quad [p, q] \mapsto [tp, tq]$$

conical action

Claim: symplectic resolution Y exists in this case

Construction ← via Slodowy varieties

Slodowy slices/varieties

$$\begin{tikzcd}[column sep=small] \mathfrak{g} \text{ — simple Lie alg.} / \mathbb{C} \quad (\mathfrak{sl}_n) \arrow[d, rightsquigarrow] & \\ \mathcal{N} := \{ x \in \mathfrak{g} \mid \mathrm{ad}_x \text{ — nilpotent} \} \arrow[r, hook, "\text{closed } G\text{-invariant}"'] & \mathfrak{g} \end{tikzcd}$$

Let us now define the resolution of $\mathcal{N}$

G — Lie group s.t. $\mathfrak{g} = \mathrm{Lie}\, G$

$B \subset G$ Borel subgrp. (maximal solvable connected subgroup of G)

$$\left(B = g \begin{pmatrix} 0 & \ddots & \ast \\ 0 & & \ddots \\ 0 & 0 \quad 0 & \end{pmatrix} g^{-1} \right)$$

$\mathfrak{b} := \mathrm{Lie}\, B$ — Borel subalgebra

$$\left(\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha \Rightarrow \mathfrak{b} = \mathrm{Ad}_g \cdot \left(\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha \right) \right)$$

flag variety
(smooth, projective)

$$\begin{tikzcd} B \arrow[r, mapsto] \arrow[d, phantom, "\in" rotate=90] & \mathrm{Lie}\, B \arrow[d, phantom, "\in" rotate=90] & \\ \{\text{Borel subgrps}\} \arrow[r, phantom, "\simeq"] & \{\text{Borel subalgebras}\} \arrow[r, phantom, "="] & G/B_0 \\ G \arrow[u, phantom, "\circlearrowright"] & G \arrow[u, phantom, "\circlearrowright"] & \end{tikzcd}$$

fixed Borel

action is transitive,
stabilizer of B_0 is $B_0 \subset G$ $\left(\text{i.e. } N_G(B_0) = B_0\right)$

Ex: $G = SL_n$, $B_0 = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix}$

$$G/B_0 \simeq \left\{ \{0\} = F_0 \subset F_1 \subset \dots \subset F_n = \mathbb{C}^n \;\middle|\; \dim F_i = i \right\}$$

$$\begin{tikzcd} G/B_0 \arrow[d, phantom, "\cup" rotate=90] \\ {[g]} \arrow[r, mapsto] & \{0\} \subset g\cdot\langle e_1\rangle \subset g\cdot\langle e_1, e_2\rangle \subset \dots \subset \mathbb{C}^n \end{tikzcd}$$

$$\tilde{\mathcal{N}} := T^*\mathcal{B} \leftarrow \text{smooth}$$

$(x,y) := \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$

$(\, , \,)$

$$T^*_{\mathfrak{b}}\mathcal{B} = (\mathfrak{g}/\mathfrak{b})^* = \mathfrak{b}^{\perp} \subset \mathfrak{g}^* \simeq \mathfrak{g}$$

nilpotent radical of $\mathfrak{b}$

$$\mathfrak{b}^{\perp} \xleftarrow{\sim} \mathfrak{n} = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}, \quad \mathfrak{n} \subset \mathfrak{g}$$

Exercise

$\Downarrow$

$$T^*\mathcal{B} = \left\{ (\mathfrak{b}, x) \;\middle|\; x \in \mathfrak{n} \right\}$$

Springer resolution
(projective since $\mathcal{B}$ is)

$$\begin{tikzcd}
T^*\mathcal{B} \arrow[r] & \mathcal{N} \\
(\mathfrak{b}, x) \arrow[u, phantom, sloped, "\in"] \arrow[r, mapsto] & x \arrow[u, phantom, sloped, "\in"]
\end{tikzcd}$$

Lem $\pi \leftarrow$ surjective

standard fact in type A follows from JNF

Prf. $\forall x \in \mathfrak{g} \ \exists \mathfrak{b} \subset \mathfrak{g}$ s.t. $x \in \mathfrak{b}$

$x \in \mathcal{N} \Rightarrow x \in \mathcal{N} \cap \mathfrak{b} = \mathfrak{n}$

Cor: $\mathcal{N} \leftarrow$ irreducible

prf: image of irreducible is irreducible

Prop: $G \circlearrowright \mathcal{N} \leftarrow$ finitely many orbits

prf. Type A $\Leftarrow$ JNF

general $\mathfrak{g} \Leftarrow$ Jacobson-Morozov thm (see below)

(will appear in homework)

Cor. $\exists! \ \mathbb{O}_{reg} \subset \mathcal{N}$ open, dense prf. G-orbits $\mathbb{O} \subset \mathcal{N}$ are loc. closed

$\dim(\mathbb{O}_1 \cup \mathbb{O}_2) \leq \max(\dim \mathbb{O}_1, \dim \mathbb{O}_2)$ so

there must be one of max dimension

$\Downarrow$

it must be open and unique

Ex. $\mathbb{O}_{reg} = SL_n \cdot \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

Prop: $\pi^{-1}(\mathbb{O}_{reg}) \xrightarrow{\sim} \mathbb{O}_{reg}$

Type A $e = \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$ preserves flag:

$$\begin{tikzcd} \{0\} \arrow[r, phantom, "\subset"] & F_1 \arrow[l, bend right] \arrow[r, phantom, "\subset"] & F_2 \arrow[l, bend right] \arrow[r, phantom, "\subset \dots \subset"] & F_{n-1} \arrow[r, phantom, "\subset"] & \mathbb{C}^n \arrow[l, bend right] \end{tikzcd}$$

$\Downarrow$

$F_i \subset \ker e^i$ ↖ already i-dim $\Rightarrow$ $F_i = \ker e^i$

Prop. $\mathcal{N} = \{x \in \mathfrak{g} \mid f(x) = 0 \ \forall f \in \mathbb{C}[\mathfrak{g}]^G_{+}\}$

$+$ means that $f(0) = 0$

Prf: 1) $f \in \mathbb{C}[\mathfrak{g}]^G_+ \multimapinv$ invariant on closures of G-orbits

2) $\mathcal{N} = \overline{\mathbb{O}_{reg}} \Rightarrow f$ − constant on $\mathcal{N}$, $f(0)=0 \Rightarrow f \equiv 0$

3) Assume $f(x)=0 \quad \forall f \in \mathbb{C}[\mathfrak{g}]^G_+$

$$\Downarrow$$

$$\det(\lambda - ad_x) = \lambda^n \Rightarrow ad_x \multimapinv \text{nilpotent}$$

$$\Downarrow$$

x − nilpotent

Cartan

$$\overset{\text{dim } \mathfrak{h}}{}$$

Now recall: $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[\Phi_i \mid i=1,\dots,r]$

Chevalley restriction thm.

$tr(x^{i+1})$ for $\mathfrak{g} = sl_{r+1}$

$$\mathcal{N} = \{x \in \mathfrak{g} \mid \Phi_i(x) = 0,\ i=1,\dots,\underline{\underline{r}}\} \curvearrowright$$

$$\begin{tikzcd} \mathcal{N} \arrow[d, hook] \\ \mathfrak{g} \end{tikzcd}$$

← complete intersection

(to check that $\mathcal{N}$ - reduced, need to prove that $d\Phi_i$ are lin. indep. at every point of $\mathbb{O}_{reg}$ $\multimapinv$ actually only at these points)

$\dim \mathcal{N} =$
$= \dim \tilde{\mathcal{N}} =$
$= 2\dim \mathfrak{n} =$
$= \dim \mathfrak{g} - \underline{\underline{r}}$

$$\Downarrow$$

$\mathcal{N} \multimapinv$ Cohen-Macaulay

$\mathcal{N}^{reg} = \mathbb{O}_{reg}$, other orbits are even-dim
(they are symplectic)
will see later

$\Downarrow$

$\mathcal{N}$ — CM and smooth in codim ≥ 2

$\Downarrow$

$\mathcal{N}$ - normal

—

Upshot:

smooth   resol. of sing.   affine, normal

$$\begin{tikzcd}
\tilde{\mathcal{N}} \arrow[r, "\pi"] & \mathcal{N} \\
\tilde{\mathbb{O}}_{reg} \arrow[u, phantom, "\cup"] \arrow[r, "\sim"] & \mathbb{O}_{reg} \arrow[u, phantom, "\cup"]
\end{tikzcd}$$

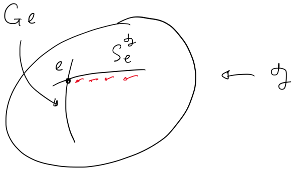
$e \in \mathcal{N}$ — nilpotent

Thm [Jacobson–Morozov]

$\exists\, h, f \in \mathfrak{g}$ s.t. $\langle \underset{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}{\underset{\uparrow}{e}}, \underset{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}{\underset{\uparrow}{h}}, \underset{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}{\underset{\uparrow}{f}} \rangle$ — form an $\mathfrak{sl}_2$-triple

$$\begin{cases} [h,e] = 2e \\ [h,f] = -2f \\ [e,f] = h \end{cases}$$

Def. $$S_e := \underbrace{\left(e + \overset{\text{centralizer of } f \text{ in } \mathfrak{g}}{\mathfrak{z}_{\mathfrak{g}}(f)}\right)}_{\text{Slodowy slice will denote by } S_e^{\mathfrak{g}}} \cap \mathcal{N} = X$$



We start with contracting $\mathbb{C}^\times$-action:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i \; ; \quad \mathfrak{g}_i = \{x \in \mathfrak{g} \mid [h,x] = ix\}$$

$(e \in \mathfrak{g}_2)$

$$t.x := t^{2-i} x \quad \text{for } x \in \mathfrak{g}_i \quad \left(t.x = t^2 \cdot (t^h x t^{-h})\right)$$

Claim: $\mathbb{C}^\times$ contracts $S_e^{\mathfrak{g}}$ to $\{e\}$

prf. $$\underset{\text{lowest vectors}}{\mathfrak{z}_{\mathfrak{g}}(f)} \subset \bigoplus_{i \leq 0} \mathfrak{g}_i$$

for $\mathfrak{sl}_2 \curvearrowright \mathfrak{g}$

intersection is transversal

Prop: (a) $S_e^{\mathfrak{g}} \cap \mathbb{O}_e = \{e\}$

(b) $T_s S_e^{\mathfrak{g}} + T_s G_s = \mathfrak{g} \quad \forall s \in S_e^{\mathfrak{g}}$

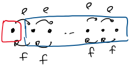
(c) $G \times S_e^{\mathfrak{g}} \to \mathfrak{g}$ ← smooth! (*)

prf. $T_e S_e^{\mathfrak{g}} = \ker(\mathrm{ad}\, f)$

$T_e \mathbb{O}_e = \mathrm{im}(\mathrm{ad}\, e)$

$$\boxed{\ker(\mathrm{ad}\, f)} \oplus \boxed{\mathrm{im}(\mathrm{ad}\, e)} = \mathfrak{g}$$

$\Uparrow$ rep. theory of $\mathfrak{sl}_2$



$\Downarrow$

$\mathbb{O}_e$ and $S_e^{\mathfrak{g}}$ are transversal at e

$\Downarrow$

use implicit function thm. to find such U

$\exists U \subset \mathfrak{g}$ s.t. $U \overset{\varphi}{\simeq} (U \cap \mathbb{O}_e) \times (U \cap S_e^{\mathfrak{g}})$

open in complex-anal. top.

$U \cap S_e^{\mathfrak{g}} \overset{\mathrm{id}}{\simeq} \{e\} \times (U \cap S_e^{\mathfrak{g}})$

$U \cap \mathbb{O}_e \overset{\mathrm{id}}{\simeq} (U \cap \mathbb{O}_e) \times \{e\}$

restrictions of φ are id

$\Downarrow$

$e \in S_e^{\mathfrak{g}} \cap \mathbb{O}_e$

isolated

$\circlearrowleft$ $\mathbb{C}^\times$ contracts to e

$\Downarrow$

$S_e^{\mathfrak{g}} \cap \mathbb{O}_e = \{e\}$

(b) follows from (a) and the existence of contracting action

(c) $G \times S_e^{\mathfrak{g}} \to \mathfrak{g}$ — differential is surjective at every point (by (b)) $\Rightarrow$ $\Rightarrow$ smooth

use contracting action

Rmk. $S_e \cap \mathbb{O}_{e'} \neq \emptyset \Rightarrow \mathbb{O}_e \subset \overline{\mathbb{O}_{e'}} \ (e \leq e')$

$$\tilde{S}_e \text{ — Slodowy variety}$$

$$\begin{tikzcd}
\tilde{S}_e \arrow[d, phantom, "\parallel"] & \\
Y := \pi^{-1}(S_e) \arrow[r, "\pi"] & S_e = X
\end{tikzcd}$$

Prop: (a) $\tilde{S}_e$ — smooth, π — resol. of sing.

(b) S_e — normal

Prf. $(*)$ $\Rightarrow$ $a, \tilde{a}$ are smooth (base change)

$$\begin{tikzcd}
G \times \tilde{S}_e \arrow[r, "\tilde{a}"] \arrow[d, "id \times \pi"] & \tilde{\mathcal{N}} \arrow[d, "\pi"] \\
G \times S_e \arrow[r, "a"] & \mathcal{N}
\end{tikzcd}$$

1) $$\begin{tikzcd} G \times \tilde{S}_e \arrow[r, "\text{smooth}"'] & \tilde{\mathcal{N}} \arrow[r, "\text{smooth}"'] & pt \end{tikzcd} \Rightarrow \tilde{S}_e\text{-smooth}$$

2) $a\left((G \times S)^{reg}\right) \subset \mathcal{N}^{reg} = \mathbb{O}_{reg} \Rightarrow$

$\Rightarrow id \times \pi$ - birational ✓

3) a - smooth $\Rightarrow$ normal $\Rightarrow$ $G \times S$ - normal

$\Downarrow$

S - normal

Rem: Only used three properties of $S_e^{\mathfrak{g}}$:

$S_e^{\mathfrak{g}}$
- smooth
- transversal to $\mathbb{O}_e$ at $\{e\}$
- $\exists\, \mathbb{C}^\times \curvearrowright S_e^{\mathfrak{g}}$ contracting it to $\{e\}$

Prop. above holds for any S with these properties

Next two lectures: $\mathbb{A}^2/\Gamma$ and its resolution as Slodowy slices / varieties, geometry of the resolution.

First example of 3d dual varieties.

$$\left(\mathbb{A}^2/\mathbb{Z}/_{n}\mathbb{Z} \leftrightarrow T^*\mathbb{P}^{n-1} \right)$$

Quantizations of $T^*\mathbb{P}^{n-1}$, category $\mathcal{O}$

and irreducible objects there.

First mirror symmetry statement,

relating geometry of resolutions

of $\mathbb{A}^2/\mathbb{Z}/_{n}\mathbb{Z}$ with irreducible objects

for quantizations of $T^*\mathbb{P}^{n-1}$