Lecture 19

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Last time:

$\mathcal{W}_\mu = G[[z^{-1}]]_1 \cdot z^\mu \leftarrow$ affine scheme

$\overline{\mathcal{W}}^\lambda_\mu \hookrightarrow \mathcal{W}_\mu \leftarrow$ closed $\Rightarrow \mathbb{C}[\overline{\mathcal{W}}^\lambda_\mu] = \mathbb{C}[\mathcal{W}_\mu] / \underset{\text{will describe later}}{\text{ideal}}$

$$\mathcal{W}_\mu = G[[z^{-1}]]_1 \Big/ \underbrace{\left(G[[z^{-1}]]_1 \cap \mathrm{Ad}_{z^\mu}(G[z])\right)}_{}$$

stabilizer of μ

Set $\mathcal{G}_\mu := G[[z^{-1}]]_1 \cap \mathrm{Ad}_{z^\mu}(G[[z^{-1}]]_1)$

$$\uparrow \wr \ \exp$$

$$\left(z^{-1}\mathfrak{g}[[z^{-1}]] \cap \mathrm{Ad}_{z^\mu}(z^{-1}\mathfrak{g}[[z^{-1}]])\right)$$

Prop: $\mathcal{G}_\mu \cdot z^\mu \xrightarrow{\sim} \mathcal{W}_\mu$

free action

so have a natural "section" of $G[[z^{-1}]]_1/\mathrm{Stab}$

prf. enough to check that the action map:

$$\mathcal{G}_\mu \times \left(G[[z^{-1}]]_1 \cap \mathrm{Ad}_{z^\mu}(G[z])\right) \xrightarrow{a} G[[z^{-1}]]_1$$

is $\underline{\underline{\text{iso}}}$.

Have a $\mathbb{C}^\times$-action by loop rotation, a – $\mathbb{C}^\times$-equiv.

contracts both source and target of a to $\{1\}$.

$\Downarrow$

enough to check that $da_{(1,1)}$ – is $\underline{\underline{\text{iso}}}$,

this follows from the definitions

Ex: $\mu = 0 \Rightarrow \mathcal{W}_0 = \mathcal{G}_0 = G[[z^{-1}]]_1$

Claim: $U[[z^{-1}]]_1 \times T[[z^{-1}]]_1 \times U_-[[z^{-1}]] \xrightarrow{\sim} G[[z^{-1}]]_1$

action (Gauss decomposition)

prf. again use contracting $\mathbb{C}^\times$-action + iso on tangent spaces at 1.

$$\left(z^{-1}\mathfrak{g}[[z^{-1}]] = z^{-1}\mathfrak{n}[[z^{-1}]] \oplus z^{-1}\mathfrak{t}[[z^{-1}]] \oplus z^{-1}\mathfrak{n}_-[[z^{-1}]]\right)$$

In general: $\mathcal{G}_\mu = G[[z^{-1}]]_1 \cap z^{\mu} \cdot G[[z^{-1}]]_1 \cdot z^{-\mu}$

Note: 1) $z^{-\mu} \cdot U[[z^{-1}]]_1 \cdot z^{\mu} \subset U[[z^{-1}]]_1$

$-\mu$ antidominant

$\Downarrow$

$z^{\mu} U[[z^{-1}]]_1 z^{-\mu} \cap U[[z^{-1}]]_1 = U[[z^{-1}]]_1$

2) $z^{\mu} T[[z^{-1}]]_1 z^{-\mu} = T[[z^{-1}]]_1$

3) $z^{\mu} U_-[[z^{-1}]]_1 z^{-\mu} \cap U_-[[z^{-1}]]_1 = z^{\mu} U_-[[z^{-1}]]_1 z^{-\mu}$

So: $\mathcal{W}_\mu = \mathcal{G}_\mu \cdot z^{\mu} = \underbrace{U[[z^{-1}]]_1}_{\substack{\text{have generators} \\ e_\alpha^{(k)},\, k>0 \\ (\alpha \in \Delta_+^\vee)}} \times \underbrace{T[[z^{-1}]]_1 \cdot z^{\mu}}_{\substack{\text{generators} \\ h_{\alpha_i}^{(\ell)},\, \ell \geq -\langle \alpha_i, \mu \rangle}} \times \underbrace{U_-[[z^{-1}]]_1}_{\substack{\text{generators} \\ f_\alpha^{(k)},\, k>0 \\ (\alpha \in \Delta_+^\vee)}}$

$h_{\alpha_i}^{(-\langle \alpha_i, \mu \rangle)} = 1$

$h_{\alpha_i}^{(\ell)}\left(t(z^{-1}) z^{\mu}\right) =$

$=$ coeff of $z^{-\ell}$ in front of $\alpha_i\left(t(z^{-1}) z^{\mu}\right) =$

$= z^{\langle \alpha_i, \mu \rangle} \cdot \alpha_i\left(t(z^{-1})\right)$

first nonzero coeff in front of $z^{\langle \alpha_i, \mu \rangle}$
$\Rightarrow -\ell \leq \langle \alpha_i, \mu \rangle \Rightarrow \ell \geq -\langle \alpha_i, \mu \rangle$

$$\mathbb{C}[\mathcal{W}_\mu] = \mathbb{C}\left[e_\alpha^{(k)}, f_\alpha^{(k)}, h_{\alpha_i}^{(\ell)} \;\middle|\; \begin{array}{l} k>0 \\ \ell \geq -\langle \alpha_i, \mu \rangle \end{array} \right]$$

$\rightsquigarrow$ quantizes [FKPRW18] [KWWY12]

$\mu = 0$ get $Y(\mathfrak{g})$ in Drinfeld's realiz.

shifted Yangian

before

$$Y_\mu = \left\langle E_{\alpha_i}^{(k)}, F_{\alpha_i}^{(k)}, H_{\alpha_i}^{(\ell)} \;\middle|\; \begin{array}{l} k>0 \\ \ell \geq -\langle \alpha_i, \mu \rangle \end{array} \right\rangle$$

$[H_i^{(p)}, H_j^{(q)}] = 0$

$[E_i^{(p)}, F_j^{(q)}] = \delta_{ij} H_i^{(p+q-1)}$

$[H_i^{(p+1)}, E_j^{(q)}] - [H_i^{(p)}, E_j^{(q+1)}] = \dfrac{\alpha_i \cdot \alpha_j}{2} \left(H_i^{(p)} E_j^{(q)} + E_j^{(q)} H_i^{(p)}\right)$

$[H_i^{(p+1)}, F_j^{(q)}] - [H_i^{(p)}, F_j^{(q+1)}] = -\dfrac{\alpha_i \cdot \alpha_j}{2} \left(H_i^{(p)} F_j^{(q)} + F_j^{(q)} H_i^{(p)}\right)$

$[E_i^{(p+1)}, E_j^{(q)}] - [E_i^{(p)}, E_j^{(q+1)}] = \dfrac{\alpha_i \cdot \alpha_j}{2} \left(E_i^{(p)} E_j^{(q)} + E_j^{(q)} E_i^{(p)}\right)$

$[F_i^{(p+1)}, F_j^{(q)}] - [F_i^{(p)}, F_j^{(q+1)}] = -\dfrac{\alpha_i \cdot \alpha_j}{2} \left(F_i^{(p)} F_j^{(q)} + F_j^{(q)} F_i^{(p)}\right)$

$i \neq j$, $N = 1 - \alpha_i \cdot \alpha_j \Rightarrow \mathrm{sym}\, [E_i^{(p_1)}, [E_i^{(p_2)}, \dots [E_i^{(p_N)}, E_j^{(q)}] \dots]] = 0$

$i \neq j$, $N = 1 - \alpha_i \alpha_j \Rightarrow \mathrm{sym}\, [F_i^{(p_1)}, [F_i^{(p_2)}, \dots [F_i^{(p_N)}, F_j^{(q)}] \dots]] = 0$

Remarks: relations in Y_μ do not depend on μ!

They are the same as for Y_0 (due to Drinfeld back in 1987)

Classical limits of these formulas define Poisson structure on W_μ

(can also extract it from Manin triple $(G[[z^{-1}]]_1, G[z], G((z^{-1})))$)

$H_i^{(\ell)}$ ← form maximal commutative subalgebra of Y_μ

$\wr$

$\mathbb{C}[T[[z^{-1}]]_1 z^\mu] \subset \mathbb{C}[W_\mu]$ ← integrable system

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Finally, let's describe $\overline{W}^\lambda_\mu$:

$$\overline{W}^\lambda_\mu = \left\{ g \in \mathcal{G}_\mu \cdot z^\mu \subset G((z^{-1})) \text{ s.t. } [g] \in \underset{\overline{G[z]\cdot z^\lambda} \subset G((z^{-1}))/G[z]}{\overline{\mathrm{Gr}^\lambda_G}} \right\}$$

We see that:

scheme

ind-scheme of ind-infinite type

$$\overline{W}^\lambda_\mu = \mathcal{G}_\mu \cdot z^\mu \cap \overline{G[z]z^\lambda G[z]} \subset G((z^{-1}))$$

ind-scheme of ind-finite type

Poisson bracket on $G((z^{-1}))$ in these coordinates:

$$\{\Delta^{(r+1)}_{\beta_1,v_1}, \Delta^{(s)}_{\beta_2,v_2}\} - \{\Delta^{(r)}_{\beta_1,v_1}, \Delta^{(s+1)}_{\beta_2,v_2}\}$$

$$\|$$

$$\sum_{a=1}^{\dim\mathfrak{g}} \Delta^{(r)}_{\beta_1,x_a v_1}\Delta^{(s)}_{\beta_2,x^a v_2} - \Delta^{(r)}_{x_a\beta_1,v_1}\Delta^{(s)}_{x^a\beta_2,v_2}$$

x_a, x^a - dual basis in $\mathfrak{g}$

For $G \curvearrowright V$, $v \in V$, $\beta \in V^*$, $s \in \mathbb{Z}$, let:

$\Delta^{(s)}_{\beta,v} \in \mathbb{C}[G((z^{-1}))]$ s.t.

$g(z) \longmapsto$ coeff. in front of z^{-s} in $\langle g\beta, v\rangle$

$\eta^{*,\vee} := -w_0\eta^\vee$

(Ex)

Claim: $$\overline{G[z]z^\lambda G[z]} = \left\{ g \in G((z^{-1})) \;\middle|\; \begin{array}{l} \Delta^{(s)}_{\beta,v}(g) = 0, \quad s > \langle \lambda, \eta^{*,\vee}\rangle \\ V = V_{\eta^\vee} \text{ ← dominant} \end{array} \right\}$$

enough to consider $\eta^\vee = \omega_i$ ← fundamental

known in type A, conjectured in general

$$\mathbb{C}[\overline{W}^\lambda_\mu] = \mathbb{C}\left[e_i^{(k)}, f_i^{(k)}, h_i^{(\ell)} \;\middle|\; \begin{array}{l} k>0 \\ \ell > -\langle \mu, \alpha_i\rangle \end{array}\right] \Big/ \overbrace{\left(\Delta^{(s)}_{\beta,v} \;\middle|\; \begin{array}{l} V = V_{\omega_i} \\ s > \langle \lambda, \omega_i^*\rangle \end{array}\right)}^{I^\lambda_\mu}$$

Gauss decomp. matrix coeff

matrix coeff.

Prop: as Poisson ideal, I^λ_μ is generated by

$\Delta^{(s)}_{v^*_{\omega_i}, v_{\omega_i}}$, $s > \langle \lambda, \omega_i^* \rangle$

already elements of $\underline{\underline{\mathbb{C}[T[[z^{-1}]]_1 z^\mu]}}$!

$$\Delta_{v^*_{\omega_i}, v_{\omega_i}}(u t u_-) = \langle (u t u_-) v^*_{\omega_i}, v_{\omega_i} \rangle =$$

$$= \langle t \cdot u_- \cdot v^*_{\omega_i}, u^{-1} v_{\omega_i} \rangle = \langle t v^*_{\omega_i}, v_{\omega_i} \rangle$$

Moreover: $a_i^{(s)} := \Delta^{(s)}_{v^*_{\omega_i}, v_{\omega_i}}$ ← different collection of generators of $\mathbb{C}[T[[z^{-1}]]_1 z^\mu]$

generate Poisson comm. alg.

quant. param.

Story quantizes: $\Delta^{(s)}_{v^*_{\omega_i}, v_{\omega_i}} \rightsquigarrow A_i^{(s)}(R)$

maybe later

$$Y^\lambda_\mu = Y_\mu / (A_i^{(s)}(R) \mid s > \langle \lambda, \omega_i \rangle)$$

So: $\overline{\mathcal{W}}^\lambda_\mu = U[[z^{-1}]]_1 T[[z^{-1}]]_1 z^\mu U_-[[z^{-1}]]_1 \cap \overline{G[z] z^\lambda G[z]}$ (*)

Important observation: this definition makes sense for arbitrary μ!

(Next time → will see the geometric meaning of (*))

$\overline{\mathcal{W}}^\lambda_\mu$ ← defined by (*) is called generalized slice in affine Grassmannian.

Rmk: $\overline{\mathcal{W}}^\lambda_\mu \to Gr_G$ ← will not be an embedding if μ ← not dominant

Let's do $G = GL_2$-example:

$\lambda = (N, 0)$, $\mu = (N-k, k)$, $k \in \mathbb{Z}_{\geq 0}$

Prop: (a) $\mathbb{W}_\mu = \left\{ M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \,\middle|\, \begin{array}{l} x_{ij} \in \mathbb{C}((z^{-1})), \ \deg x_{21}, x_{12} < N \\ x_{22} = z^N + \text{lower}, \ \det M = z^N + \text{lower} \end{array} \right\}$

(b) $\overline{\mathcal{W}}^\lambda_\mu = \left\{ M = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \,\middle|\, \begin{array}{l} x_{ij} \in \mathbb{C}[z], \ \deg x_{21}, x_{12} < N \\ x_{22} = z^k + \text{lower}, \ \det M = z^N \end{array} \right\}$

prf: direct computation

$$\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} z^{N-k} & 0 \\ 0 & z^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} t_1 z^{N-k} & t_2 z^k e \\ 0 & t_2 z^k \end{pmatrix} \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} = \begin{pmatrix} t_1 z^{N-k} + t_2 z^k ef & t_2 z^k e \\ t_2 z^k f & t_2 z^k \end{pmatrix}$$

$$\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$$

See that $x_{22} = z^k t_2 \in z^k + z^{k-1}\mathbb{C}[[z^{-1}]]$

$\deg B, C \overset{Q}{<} k \quad (e, f \in z^{-1}\mathbb{C}[[z^{-1}]])$

— in the opposite direction: $t_2 = \dfrac{x_{22}}{z^k}$, $t_1 = \dfrac{\det M}{z^N t_2}$

$$e = \frac{x_{12}}{x_{22}}$$

$$f = \frac{x_{21}}{x_{22}}$$

Let's deal with (b): from our description of $\overline{Gr}^{\lambda}$ we see that the condition is exactly $x_{ij} \in \mathbb{C}[z]$ (corresponds to $V = V_{\omega_1}$) and

$\det M \in z^N \cdot \mathbb{C}[z] \Rightarrow \det M = z^N$

(corresponds to $V = V_{\omega_2}$)

Note: $a_{11} = x_{22}$; $a_{22} = x_{11} x_{22} z^N$

Examples (see lecture 1!)

$$\overline{\mathcal{W}}^{(N,0)}_{(N-1,1)} = \left\{ \begin{pmatrix} \frac{z^N - a^N}{z-a} & b \\ c & z-a \end{pmatrix} \middle| \; a^N + bc = 0 \right\} = \mathbb{A}^2/\mathbb{Z}/N\mathbb{Z}$$

Note: $N \geq 2 \Rightarrow \mu \leftarrow$ dominant

$N = 1 \Rightarrow \mu = (0,1) \leftarrow$ anti-dominant

$\overline{\mathcal{W}}^{(1,0)}_{(0,1)} \simeq \mathbb{A}^2 \leftarrow$ smooth!

Another example: $N = 0 \Rightarrow \overline{\mathcal{W}}^{(0,0)}_{(-k,k)} = \{ M \text{ as above}, \det M = 1 \}$

$$\Vert$$

$$\left\{ f : \mathbb{P}^1 \to \mathbb{P}^1 \;\middle|\; \begin{matrix} \deg f = k \\ f(\infty) = \infty \end{matrix} \right\}$$

Geometric definition of $\overline{W}^\lambda_\mu$ (will allow to construct resolution)

Prop.
$$\overline{W}^\lambda_\mu = \left\{ (\mathcal{P},\sigma,\mathcal{P}^B) \;\middle|\; \begin{array}{l} (\mathcal{P},\sigma)\in Gr_G \\ \deg \mathcal{P}^B = w_0\mu \\ \underset{B_-}{\underset{\parallel}{\mathcal{P}^B_\infty}} \subset \mathcal{P}_\infty \overset{\sigma_\infty}{\simeq} G \end{array}\right\}$$

$\mathcal{P}^B \subset \mathcal{P}$

B-torsor

$\mathcal{P}$ a on $\mathbb{P}^1$

← the only condition that relates σ with $\mathcal{P}^B$

prf. ← next time

$$g \in B[[z^{-1}]]_1\, z^\mu B_-[[z^{-1}]]_1 \cap \overline{G[z]z^\lambda G[z]}$$

$\rightsquigarrow$

$\mathcal{P}$ and trivializations $\sigma_0 : \mathcal{P}^{triv}|_{\mathbb{P}^1\setminus\{0\}} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{P}^1\setminus\{0\}}$

$\sigma_\infty : \mathcal{P}^{triv}|_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{P}|_{\mathbb{A}^1}$

Now $\mathcal{P}^{B_-} :=$ image of the standard B_--structure under σ_∞

Ex: $\lambda = 0 \Rightarrow \overline{W}^0_\mu = \{ \mathcal{P}^B \subset \mathcal{P}^{triv} \} = \{ f : \mathbb{P}^1 \to G/B \}$

$\mathcal{P}^B$: B-structure of deg $w_0\mu$

f: deg $= w_0\mu$, $\infty \mapsto B_-$

Zastava spaces

(B-structure in G-bundle $\begin{matrix}\mathcal{E}\\ \downarrow \\ X\end{matrix}$ is the same as section $X \to \mathcal{E}/B$)

if $\mathcal{E} \simeq G\times X \Rightarrow$ consider sections $X \to (G/B)\times X \Leftrightarrow X \to G/B$

How physicists call $\overline{W}^\lambda_\mu$ ← moduli space of (singular) G-monopoles on $\mathbb{R}^3 = \mathbb{R}\times\mathbb{C}$

(μ ← singular monopole charge)

(λ ← total charge)

Our "matrix" description is what physicists call "scattering matrix" description of monopoles

Physical observations helped to define multiplication morphism

$\overline{W}^\lambda_\mu \times \overline{W}^{\lambda'}_{\mu'} \to \overline{W}^{\lambda+\lambda'}_{\mu+\mu'}$ ← will discuss next time, this is the key thing to understand $\overline{W}^\lambda_\mu$'s

Next time

"monopoles on $\mathbb{R}^3$..."

How to think about $\overline{\mathcal{W}}^{\lambda}_{\mu}$ more "geometrically"

$$\mathcal{W}_\mu = G[z^{-1}]_1 \cdot z^\mu = G[z^{-1}]_{B_-} \cdot z^\mu$$

(use that $B_- \cdot z^\mu = z^\mu$ as $B_- \subset P_\mu$)

Now, $G[z^{-1}] \cdot z^\mu = \{ (\mathcal{P}, \sigma) \mid \mathcal{P} \simeq \mathcal{O}(z^\mu) \}$

$$G[z^{-1}]_{B_-} \cdot z^\mu = \left\{ (\mathcal{P},\sigma) \;\middle|\; \begin{array}{l} \mathcal{P} \overset{\exists}{\simeq} \mathcal{O}(z^\mu) \\ \begin{tikzcd} \mathcal{P}^{\mathrm{triv}}_\infty \arrow[r, "\sigma_\infty", "\simeq"'] & \mathcal{P}_\infty \arrow[r, "\simeq"] & \mathcal{O}(z^\mu)|_\infty \arrow[r, "\sigma_{z^\mu}", "\simeq"'] & \mathcal{P}^{\mathrm{triv}}_\infty \\ G \arrow[u, equal] & & & G \arrow[u, equal] \\ B_- \arrow[u, hook] \arrow[rrr, mapsto] & & & B_- \arrow[u, hook] \end{tikzcd} \end{array} \right\}$$

Claim: if $(\mathcal{P},\sigma) \in \mathcal{W}_\mu \Rightarrow \exists!\, \mathcal{P}^B \subset \mathcal{P}$

deg $= w_0\mu$ B-subtorsor (also called B-structure)

s.t. $\mathcal{P}^B|_\infty \subset \mathcal{P}_\infty \overset{\sigma}{=} G$

is B_-

prf. Do $G = GL_n$ (for arbitrary G should consider Harder-Narasimhan filtration)

So $\mathcal{P} = \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_n)$, $\underbrace{k_1 \geq k_2 \geq \dots \geq k_n}_{\mu}$

B-structure of deg $= w_0\mu$ is:

$$\{0\} \overset{\mathcal{O}(k_1)}{\subset} \mathcal{F}_1 \overset{\mathcal{O}(k_2)}{\subset} \mathcal{F}_2 \overset{\dots}{\subset} \dots \subset \mathcal{F}_{n-1} \overset{\mathcal{O}(k_n)}{\subset} \mathcal{F}_n = \mathcal{P}$$

$(\mathcal{F}_i / \mathcal{F}_{i-1} \simeq \mathcal{O}(k_i))$

at ∞ should be equal to...

Prop: $$\overline{\mathcal{W}}_\mu = \left\{ (\mathcal{P},\sigma,\mathcal{P}^B) \;\middle|\; \begin{array}{l} (\mathcal{P},\sigma) \in Gr_G \\ \mathcal{P}^B \text{ a B-structure in } \mathcal{P} \\ \deg \mathcal{P}^B = w_0\mu \\ \mathcal{P}^B|_\infty = B_- \end{array} \right\}$$

Resolutions

say about Coulomb and monopoles

Multiplication of slices